Essays on Empirical Macroeconomics

Dario Caldara
Abstract

What Are the Effects of Fiscal Policy Shocks? A VAR-Based Comparative Analysis. The empirical literature using vector autoregressive models to assess the effects of fiscal policy shocks strongly disagrees on the qualitative response of key macroeconomic variables to government spending and tax shocks. We provide new evidence for the U.S. over the period 1955-2006. We show that, controlling for differences in specification of the reduced-form model, all identification approaches used in the literature yield qualitatively and quantitatively very similar results as regards government spending shocks. In response to such shocks, there is a significant increase in real GDP, real private consumption and the real wage following a hump-shaped pattern, while there is no reaction in private employment. In contrast, we find strongly diverging results as regards the effects of tax shocks, with the estimated effects ranging from non-distortionary to strongly distortionary. The differences in results can to a large extent be traced back to differences in the size of automatic stabilizers estimated or calibrated for alternative identification approaches. These differences also translate into uncertainty about the effects of policy experiments typically considered in theoretical models.

The Analytics of SVARs: A Unified Framework to Measure Fiscal Multipliers. The empirical literature using vector autoregressive models to assess the effects of fiscal policy shocks strongly disagrees on the qualitative response of key macroeconomic variables to government spending and tax shocks. We provide new evidence for the U.S. over the period 1955-2006. We show that, controlling for differences in specification of the reduced-
form model, all identification approaches used in the literature yield qualita-
tively and quantitatively very similar results as regards government spending
shocks. In response to such shocks, there is a significant increase in real GDP,
real private consumption and the real wage following a hump-shaped pat-
tern, while there is no reaction in private employment. In contrast, we find
strongly diverging results as regards the effects of tax shocks, with the esti-
mated effects ranging from non-distortionary to strongly distortionary. The
differences in results can to a large extent be traced back to differences in
the size of automatic stabilizers estimated or calibrated for alternative iden-
tification approaches. These differences also translate into uncertainty about
the effects of policy experiments typically considered in theoretical models.

Computing DSGE Models with Recursive Preferences and Sto-
chastic Volatility. This paper compares different solution methods for com-
puting the equilibrium of dynamic stochastic general equilibrium (DSGE)
models with recursive preferences such as those in Epstein and Zin (1989,
1991) and stochastic volatility. Models with these two features have recently
become popular, but we know little about the best ways of implementing
them numerically. To fill this gap, we solve the stochastic neoclassical growth
model with recursive preferences and stochastic volatility using four different
approaches: second- and third-order perturbation, Chebyshev polynomials,
and value function iteration. We document the performance of the methods
in terms of computing time, implementation complexity, and accuracy. Our
main finding is that perturbations are competitive in terms of accuracy with
Chebyshev polynomials and value function iteration, and several orders of
magnitude faster to run. Therefore, we conclude that perturbation methods
are an attractive approach for computing this class of problems.

Business Cycle Accounting and Misspecified DSGE Models. In
this paper we consider how insights from a range of models can be used
to trace out the implications of ‘missing channels’ in a baseline estimated
dynamic stochastic general equilibrium (DSGE) model used for forecast and policy analysis. Specifically, we show how the insights from the business cycle accounting (BCA) methodology may be applied to the issue of misspecification in DSGE models. A key insight from BCA analysis is that shocks or transmission channels that are not captured in the baseline DSGE model may appear as correlated ‘shocks’ to that model. We argue that it may be possible to map from missing channels to structural shocks by applying BCA insights to the baseline DSGE model. The key idea is to introduce a proxy variable that captures the effects of a missing channel and relate the innovations to this proxy variable to a (small) set of atheoretical ‘factors’. We then allow these factors to feed into the structural shocks of the DSGE model to create correlated movements in those shocks.
To Ania
viii
Acknowledgments

During the past six years I have had the opportunity to meet and interact with many extraordinary people and their influence has been essential to the ideas developed in this thesis. First of all I would like to thank my advisors John Hassler and Jesús Fernández-Villaverde for their guidance, thoughtful critic, and encouragement. I am also thankful to Torsten Persson and Frank Schorfheide, who provided invaluable insights on chapter 3 of this thesis. My interest in fiscal policy and structural vector autoregressions is due largely to the welcome influence of my friend and co-author of chapter 2, Christophe Kamps. I thank Yao Wen for making the long evenings of Fortran coding needed to complete chapter 4, more enjoyable. I am grateful to Richard Harrison for bringing me on board at the Bank of England Quarterly Model review project during my internship at the Bank of England. Our discussions became the subject of Chapter 5. Galina Hale and Ethan Kaplan introduced me to the world of empirical microeconometrics and political business cycles. It is a pity our joint work cannot be part of my thesis, but I hope that living in the same country will help us to bring the paper to a successful completion.

The first year of my PhD would have been much less exciting without the friendship, parties, and 7th-floor dinners with Shon Ferguson, Johan Gars, Marta Lachowska, Joachim Nilsson, Carin Kuylenstjerna, David Yanagizawa, and Acke Wenelius.

The IIES provided a stimulating research environment, and I enjoyed interacting with all faculty and graduate students. In particular, I thank Fabrizio Zilibotti for introducing me to the idea of doing a PhD, and for his support thereafter. I started at the IIES as a research assistant, and I was lucky to have David von Below, Erik Meyersson, Ettore Panetti, and Jinfeng as colleagues. I thank Erika Färnstrand-Damsgaard, Daria Finocchiaro, Andreas Mueller, Maria Perrotta, Ettore Panetti, Daniel Spiro, and Mirco Tonin for discussing my research ideas, for sharing my joys and frustrations, and for being such good friends. I would have been lost without the administrative
staff of the IIES. Annika Andreasson answered thousands of questions and provided moral support during the job market. Christina Lönblad helped me travel around the world and provided great editorial support. I also gratefully acknowledge the financial support provided by Jan Wallander and Tom Hedelius Research Foundations. Outside the IIES, I enjoyed the friendship of David D’Angelo, Nick Sheard, Mark Sanctuary, Margherita Bottero, and Sara Formai, to mention just a few.

I am very thankful to the faculty and students at the University of Pennsylvania, who adopted me for two academic years. In particular, Luigi Bocola, Cristina Fuentes-Albero, Leonardo Melosi, and Max Kryshko provided excellent feedback on my papers and enjoyable discussions on empirical macroeconomics. I also thank Drew Griffen, Fatih Karahan, Nirav Mehta, Kathleen Molnar, Antonio Penta, Gil Shapira, Panos Stavrinides, all visiting students, officemates, and roommates for making me feel at home in Philadelphia, and for coming to all my farewell parties.

I would like to thank the Fiscal Policies and the Monetary Strategy divisions at the European Central Bank for hosting me in several occasions, and for the feedback on my research on Structural VARs. I am thankful to James Proudman and Tony Yates for their kind hospitality in the Monetary Analysis Strategy Division at the Bank of England. The idea for Chapter 3 of this thesis originated while being an Intern at the Sveriges Riksbank, where I enjoyed talking to Tor Jacobson, Ulf Söderström, Daria Finocchiaro, Virginia Queijo von Heideken, and Matthias Villani.

Throughout this journey I always enjoyed the love and support of my parents Antonietta and Glauco, and my sister Claudia. My greatest gratitude goes to my wife, Anna Lipińska. She stood beside me through the toughest moments and supported me during those times I doubted that I could successfully complete this journey. She has been always supportive of my choices, even when they implied sacrifices on her part. Kochanie, I dedicate this thesis to you.
Table of Contents

1 Introduction 1
   References ........................................ 8

2 What Are the Effects of Fiscal Policy Shocks? A VAR-Based
   Comparative Analysis 9
   2.1 Introduction ....................................... 9
   2.2 Data .............................................. 15
   2.3 Econometric Methodology ........................... 17
   2.4 Results for the Pure Fiscal Shocks ................. 26
   2.5 The Size of Automatic Stabilizers .................... 30
   2.6 Results for the Policy Experiments ................... 33
   2.7 Robustness ....................................... 36
   2.8 Conclusions ...................................... 39
   References ........................................ 44
   A Data Appendix ..................................... 44
   B Figures ....................................... 48

3 The Analytics of SVARs: A Unified Framework to Measure
   Fiscal Multipliers 59
   3.1 Introduction ....................................... 59
   3.2 The Econometric Framework ....................... 64
   3.3 The Analytics of Identification ................... 66
# TABLE OF CONTENTS

3.4 The Literature Road Map ........................................... 76  
3.5 Deriving Restrictions on Elasticities ............................ 87  
3.6 Extensions ................................................................ 95  
3.7 Conclusions ............................................................. 98  
References .................................................................... 104  
A Appendix .................................................................... 104  

4 Computing DSGE Models with Recursive Preferences and  
Stochastic Volatility .................................................... 135  
4.1 Introduction .............................................................. 135  
4.2 The Model ................................................................ 138  
4.3 Solution Methods ....................................................... 142  
4.4 Calibration ............................................................... 158  
4.5 Numerical Results ...................................................... 159  
4.6 Conclusions ............................................................. 168  
References .................................................................... 173  
A Appendix .................................................................... 174  

5 Business Cycle Accounting and Misspecified DSGE Models 187  
5.1 Introduction .............................................................. 187  
5.2 A Sketch of the Idea ................................................... 189  
5.3 Two Simple Examples ................................................ 200  
5.4 Empirical Application ................................................ 214  
5.5 Conclusions ............................................................. 220  
References .................................................................... 224  
A Appendix .................................................................... 224
Chapter 1

Introduction

This thesis consists of four self-contained essays that deal with different aspects of empirical macroeconomics. The first two essays employ structural vector autoregressions (SVARs) to study the effects of changes in taxation and public expenditures in the United States. The third essay compares different numerical techniques to solve dynamic stochastic general equilibrium (DSGE) models. The fourth essay proposes a method to trace the implications of missing channels in a baseline estimated DSGE model.

SVARs have been introduced by Sims (1980), and have quickly become a standard tool of modern macroeconometric analysis. SVARs are a multivariate, linear representation of a vector of observables on its own lags, and are used by economists to recover economic shocks from observables by imposing a minimum of assumptions compatible with a large class of models (Fernández-Villaverde and Rubio-Ramírez, 2008).

DSGE models have become the workhorse of theoretical and empirical macroeconomics thanks to the seminal work of Kydland and Prescott (1982). DSGE models impose tight assumptions regarding the structure of the economy (utility functions, production functions, market clearing conditions) and the behavior of agents, which usually are assumed to form expectations rationally.
While the use of DSGE models for policy analysis has become uncontentious, in recent years there has been some debate regarding the extent to which one could derive policy prescriptions from SVARs. In particular, some authors describe SVARs as reduced form models, subject to the Lucas’ critique. Instead, SVARs are “structural” in the sense of Hurwicz (1966), that is, they are invariant to the policy under investigation. This confusion arises because it has become common to refer to DSGE models as structural models, regardless of whether they satisfy the definition of Hurwicz. On the other hand, models that are structural in the old sense, as SVARs, are now thought to be reduced form, because their parameters do not have a unique behavioral interpretation.

In what follows I briefly summarize the content and results of each chapter.

Governments often use fiscal policy to stabilize economic fluctuations. For example, during the recent recession, the United States Congress approved the American Recovery and Reinvestment Act, which introduced increases in public spending and cuts in taxes by approximately 6% of GDP. The rationale for such fiscal stimulus rests on the assumption that fiscal interventions do stabilize the economy. Yet, the size of fiscal multipliers, defined as the dollar response of output to an exogenous dollar spending increase or tax cut, is the subject of a long-standing debate in academia. As Perotti (2007) observes in his survey of the literature: “... perfectly reasonable economists can and do disagree on the basic theoretical effects of fiscal policy and on the interpretation of existing empirical evidence”.

The presence of competing economic theories has motivated a large body of empirical investigations that measure the size of these fiscal multipliers. An important share of the literature relies on structural vector autoregressions (SVARs). Prominent examples include Blanchard and Perotti (2002), and Mountford and Uhlig (2009). The appeal of SVARs is that they control for endogenous movements in fiscal policies by only imposing a minimal
set of assumptions, known as identification schemes. Yet, despite their simple structure, studies employing SVARs document fiscal multipliers that are spread over a broad range of values. So far, little effort has been devoted to understanding which assumptions in competing SVARs drive differences in results. The lack of robust evidence prevents the profession from providing clear guidance on important policy choices, such as the size and composition of fiscal interventions.

Motivated by this lack of knowledge, the first two essays in my thesis share the following question: Why do SVARs provide different measures of fiscal multipliers?

**What are the Effects of Fiscal Policy Shocks? A VAR-based Comparative Analysis.** Chapter 2, co-authored with Christophe Kamps, shows that after controlling for differences in specification of the reduced-form VAR model, some of the disagreement in the literature vanishes. In particular, all identification approaches used in the literature yield qualitatively very similar results as regards the effects of government spending shocks. The evidence presented in our paper suggests that private consumption increases in response to a positive government spending shock. Our empirical results support models which generate an increase in the real wage, but at the same time do not support the increase in employment implied by most current-generation DSGE models. Furthermore, the positive responses of private consumption and the real wage are very persistent, whereas most current-generation DSGE models predict that the responses turn negative already one year after the government spending shock occurs.

In contrast, we find strongly divergent results as regards the effects of tax shocks depending on the identification approach used, with the estimated effects ranging from non-distortionary to strongly distortionary. We conjecture that the differences in results can to a large extent be traced back to differences in the size of automatic stabilizers estimated or calibrated for alternative identification approaches, with the estimated degree of distortion
associated with a given tax shock being positively related to the size of automatic stabilizers.

The Analytics of SVARs: A Unified Framework to Measure Fiscal Multipliers. Chapter 3 derives a unified analytical framework to compare competing identification approaches. First, I show analytically that existing identification schemes imply different restrictions on the output elasticity of tax revenue and government spending. These elasticities measure the endogenous response of tax and spending policies to economic activity. Then, I show that different restrictions on the output elasticity of tax revenue and government expenditures generate a large dispersion in the estimates of tax and spending multipliers.

These findings lead me to ask the following question: Can we construct robust measures of fiscal multipliers using SVARs? I propose to measure fiscal multipliers more robustly by imposing restrictions on the output elasticities of fiscal variables in the form of probability distributions. In contrast to the existing literature, I measure these distributions both by using a variety of empirical strategies and by employing a simple DSGE model. I find that the direct measurement of prior distributions reduces the dispersion of output elasticities implied by existing identification schemes. These restrictions are robust because they are generated by different approaches and empirical strategies and, hence, are less likely to be affected by particular assumptions or observations.

I apply this robust identification scheme to measure tax and spending multipliers associated with unexpected fiscal shocks. I document three findings. First, the median impact tax multiplier is close to 0. Second, the median impact spending multiplier is 0.7 and ranges between 0.35 and 1. Third, estimates of fiscal multipliers at longer horizons are dispersed over a broad range. Despite this uncertainty, the probability that the spending multiplier is larger than the tax multiplier is above 0.8, for up to four years after policy interventions.
Computing DSGE Models with Recursive Preferences and Stochastic Volatility. Chapter 4, co-authored with Jesús Fernández-Villaverde, Juan Rubio-Ramírez, and Yao Wen, compares different solution methods for computing the equilibrium of DSGE models with recursive preferences and stochastic volatility. Both features have become very popular in finance and macroeconomics as modeling devices to account for business cycle fluctuations and asset pricing. Recursive preferences, as those proposed by Epstein and Zin (1989), are attractive for two reasons. First, they allow us to separate risk aversion and intertemporal elasticity of substitution (EIS). Second, they offer the intuitive appeal of having preferences for an early or later resolution of uncertainty. Stochastic volatility generates aggregate fluctuations with time-varying volatility, a basic property of many time series, and adds extra flexibility in accounting for asset pricing patterns.

Despite the popularity of these issues, little is known about the numerical properties of the different solution methods that solve equilibrium models with recursive preferences and stochastic volatility. Importantly, the most common solution algorithm in the DSGE literature, (log-) linearization, cannot be applied. The resulting (log-) linear decision rules are certainty equivalent and do not depend on risk aversion or volatility.

We solve and simulate the model using four main approaches: perturbation (of second- and third-order), Chebyshev polynomials, and value function iteration (VFI). We highlight four results. First, all methods provide a high degree of accuracy. Thus, researchers who stay within our set of solution algorithms can be confident that their quantitative answers are sound. Second, perturbations deliver a surprisingly high level of accuracy with considerable speed. Since, in practice, perturbation methods are the only computationally feasible method for solving the medium-scale DSGE models used for policy analysis that have dozens of state variables (as in Smets and Wouters, 2007), this finding has an outmost applicability. Third, Chebyshev polynomials provide a terrific level of accuracy with a reasonable computational
INTRODUCTION

When accuracy is most required and the dimensionality of the state space is not too high, they are the obvious choice. Fourth, we were disappointed by the poor performance of VFI which, compared with Chebyshev, could not achieve a high accuracy even with a large grid.

Business Cycle Accounting and Misspecified DSGE Models. Chapter 5, co-authored with Richard Harrison, proposes a method to trace out the implications of missing channels in a baseline estimated DSGE model used for forecasting and policy analysis. In the past ten years, the role of DSGE models in central banks has increased markedly. Operational central bank models are larger than their academic counterparts. One important reason is the policymakers’ desire to have detailed and comprehensive discussions about a large number of shocks and transmission channels.

All models, however large, are misspecified. For example, none of the DSGE models in operational use at central banks contain explicit modelling of financial frictions or banking. One response to the observation that operational models exclude some channels and mechanisms of interest is to expand them accordingly. Yet, large models are inherently harder to understand and explain to busy policymakers. Even if this strategy is a desirable long-term objective, in the short run it is possible that the economic issues relevant to policy discussions develop more quickly than the operational forecast models used to support those discussions.

In this essay, we consider how insights from other DSGE models can be used to trace out the implications of ‘missing channels’ in a baseline estimated DSGE model used for forecast and policy analysis. Specifically, we introduce a proxy variable that captures the effects of a missing channel and relate the innovations to this proxy variable to a set of atheoretical ‘factors’. We then allow these factors to feed into the structural shocks of the model to create correlated movements in those shocks.

We illustrate the approach using two simple examples, in which a policymaker has access to a misspecified model of the economy. Our first example
is one in which oil prices affect the supply side of the economy, but the policymaker’s model does not include a role for oil. Our second example is one in which house prices have a financial accelerator effect on demand, but the policymaker’s model does not include house prices or any mechanisms through which they may play an important role. We find that our method can successfully account for the effects of missing channels in the policymaker’s model, although this does not necessarily lead to an improvement in the forecasting performance of the model.

Bibliography


INTRODUCTION


Chapter 2

What Are the Effects of Fiscal Policy Shocks? A VAR-Based Comparative Analysis*

2.1 Introduction

In recent years, vector autoregressive (VAR) models have become the main econometric tool for assessing the effects of monetary and fiscal policy shocks. While a consensus view has emerged as regards the empirical effects of monetary policy shocks (Christiano et al., 1999), the empirical literature has so far struggled to provide robust stylized facts on the effects of fiscal policy shocks (Perotti, 2007). In particular, there is no agreement on the qualitative effects of fiscal policy shocks on those macroeconomic variables (private

*This paper is co-authored with Christophe Kamps. We would like to thank an anonymous referee, Kai Carstensen, Efrem Castelnuovo, Andrew Mountford, Torsten Persson and Harald Uhlig for helpful comments and discussions. We also thank seminar audiences at the IIES, the ECB, the Kiel Institute for the World Economy, the Universities of Padua, Pavia and Tübingen as well as at the 2006 congresses of the EEA, the Society for Computational Economics and the IIPF and at the 2007 congress of the Royal Economic Society for helpful comments. The views expressed in this paper are those of the authors and do not necessarily reflect those of the European Central Bank.
consumption, real wage and private employment) which it would be helpful to discriminate among competing theories. In this paper, we show that after controlling for differences in the specification of the reduced-form VAR model, some of the disagreement in the literature vanishes. In particular, all identification approaches used in the literature yield qualitatively and quantitatively very similar results as regards the effects of government spending shocks. In contrast, we find strongly divergent results as regards the effects of tax shocks depending on the identification approach used, with the estimated effects ranging from non-distortionary to strongly distortionary. The differences in results can to a large extent be traced back to differences in the size of automatic stabilizers estimated or calibrated for alternative identification approaches, with the estimated degree of distortion associated with a given tax shock being positively related to the size of automatic stabilizers. These differences also translate into uncertainty about the effects of policy experiments typically considered in theoretical macroeconomic models. In the case of balanced-budget spending increases, e.g., the sign of the fiscal multiplier depends on the identification approach chosen. We also provide new evidence for deficit-financed spending increases and deficit-financed tax cuts.

Apart from differences in the specification of the reduced-form VAR model (including sample period, set of endogenous variables, deterministic terms and lag length), the empirical studies in this literature distinguish themselves by the approach chosen to identify fiscal policy shocks. Four main identification approaches have been used to date: first, the recursive approach introduced by Sims (1980) and applied to study the effects of fiscal shocks by Fatas and Mihov (2001); second, the structural VAR approach proposed by Blanchard and Perotti (2002) and extended in Perotti (2005, 2007); third, the sign-restrictions approach developed by Uhlig (2005) and applied to fiscal policy analysis by Mountford and Uhlig (2009); and, fourth, the event-study approach introduced by Ramey and Shapiro (1998) to study the effects of large unexpected increases in government defense spending and also used
by Edelberg et al. (1999), Eichenbaum and Fisher (2005), Perotti (2007) and Ramey (forthcoming). In this paper, we use all four identification approaches. We first discuss the empirical evidence provided by this literature for government spending shocks as this evidence, conflicting as it may be, has strongly influenced recent theoretical modeling of fiscal policy. Before turning to the disagreement, it is interesting to note that irrespective of the chosen identification approach, all studies agree that positive government spending shocks have persistent positive output effects. However, this finding on its own is not helpful for discriminating among competing theories because a positive output response is compatible with both Keynesian and neoclassical theories.\(^1\) Yet, the empirical studies in this literature disagree on the effects of government spending shocks on those macroeconomic variables which are helpful in this respect. In particular, this is true for the response of private consumption. Fatas and Mihov (2001), Blanchard and Perotti (2002) and Perotti (2005, 2007) report that private consumption significantly and persistently increases in response to a positive government spending shock, while Mountford and Uhlig (2009) and Edelberg et al. (1999) provide evidence that the response of private consumption is close to zero and statistically insignificant over the entire impulse response horizon. Ramey (forthcoming) reports that private consumption persistently and (over short and long horizons) significantly falls in response to such a shock. As regards the responses of real wage and employment, Perotti (2007) provides evidence that the real wage persistently and significantly increases while there is no reaction in employment, whereas Eichenbaum and Fisher (2005) and Burnside et al. (2004) provide evidence that the real wage persistently and significantly falls while employment persistently and significantly increases.

The recent theoretical literature modeling the effects of fiscal policy shocks using dynamic stochastic general equilibrium (DSGE) models has evolved

\(^1\)In the case of neoclassical theories, a positive output response is only obtained if the increase in government spending is financed by non-distortionary taxes (Baxter and King, 1993).
along two very different lines in response to this empirical evidence. The first branch of this literature builds on the assumption that private consumption and the real wage respond negatively and employment positively to an increase in government spending. If those were the relevant stylized facts, (variants of) the prototypical real business cycle (RBC) model would be fully data-consistent. In this model, an exogenous increase in government spending financed by lump-sum taxes reduces the representative agent’s wealth causing the agent to consume less and work more which, in turn, depresses the real wage. Examples include Edelberg et al. (1999), Burnside et al. (2004) and Eichenbaum and Fisher (2005). The second branch of this literature, instead, takes as a stylized fact that private consumption responds positively to an increase in government spending. If this were a robust stylized fact, then the standard neoclassical model would not be data-consistent. Several authors have introduced modifications to the standard model in order to make its predictions consistent with a rise in private consumption\(^2\): Using a modified utility function for which consumption and employment are complements, Linnemann (2006) shows that for empirically plausible parameter values, private consumption and employment increase while the real wage falls in response to a positive government spending shock. Ravn et al. (2006) incorporate good-specific habits into a model with monopolistic competition and show that for large values of the habit-persistence parameter private consumption, the real wage and employment all increase in response to a government spending shock. Gali et al. (2007) incorporate rule-of-thumb consumers into a model with nominal rigidities and show that—for a sufficiently large size of the group of rule-of-thumb consumers—private consumption, the real wage and employment all increase in response to a government spending shock. The evidence presented in our paper suggests that private consumption indeed increases in response to a positive government spending shock and that the responses of labor market variables seem to be important for

\(^2\)See Perotti (2007) for a more comprehensive review of this branch of the literature.
rationalizing the consumption response. Our empirical results support models which generate an increase in the real wage but, at the same time, do not support the increase in employment implied by most current-generation DSGE models. A further challenge arising from the empirical evidence is that the positive responses of private consumption and the real wage are very persistent, whereas most current-generation DSGE models consistent with an increase in these variables predict that the responses turn negative already about one year after the government spending shock occurs (see e.g. Gali et al., 2007).

As regards tax shocks, the empirical literature is also characterized by some disagreement on their macroeconomic effects. Most studies assessing the effects of tax shocks on the U.S. economy conclude that unanticipated tax increases have strong negative effects on output and other real economy variables. This is true for studies using the sign-restrictions approach (see Mountford and Uhlig, 2009) or a narrative approach (similar to the event-study approach for government spending shocks) isolating those legislated tax changes which were unrelated to the state of the economy and using them to estimate the macroeconomic effects of exogenous tax changes (Romer and Romer, 2010). In contrast, the structural VAR approach introduced by Blanchard and Perotti (2002) and further developed by Perotti (2005) yields conflicting evidence. While Blanchard and Perotti (2002) provide evidence showing that unanticipated tax increases have strongly negative output effects, the results in Perotti (2005) suggest that there is no reaction in output in the U.S. in the period when the tax shock hits the economy.

---

3 This disagreement has received much less attention in the recent theoretical literature. For simplicity, nearly all theoretical studies assume taxes to be non-distortionary. Moreover, if taxes were instead assumed to be distortionary, it would not only be very difficult to generate a rise in private consumption in response to a tax-financed increase in government spending but also to obtain an increase in output.

4 This difference in results largely seems to be due to the different definitions of taxes used by Blanchard and Perotti (2002) and Perotti (2005). While Blanchard and Perotti (2002) use cash data on federal corporate income tax receipts from the Quarterly Treasury Bulletin, Perotti (2005) uses accrual data provided with the National Income and Product
Empirical results indicate that the answer to the question of whether taxes are distortionary or not depends on the identification approach chosen. While our results for the Blanchard-Perotti approach suggest that taxes are non-distortionary, our results for the sign-restrictions approach suggest that taxes are strongly distortionary. We further show that the answer depends strongly on the size of automatic stabilizers, which is lower for the Blanchard-Perotti approach (for which the size of automatic stabilizers is calibrated on the basis of extra-model evidence) than for the sign-restrictions approach (for which the size of automatic stabilizers is estimated inside the model). We show that for the Blanchard-Perotti approach, there is an approximately linear relationship between the calibrated size of automatic stabilizers and the estimated sign and size of the impact output response to exogenous tax shocks. We interpret our results as indicating a need for a refinement of the way in which taxes are adjusted for the effects of the business cycle in structural VAR models.

The uncertainty about the effects of tax shocks translates into uncertainty about the effects of policy experiments typically considered in the theoretical literature. We present evidence for three alternative policy experiments: a balanced-budget spending increase, a deficit-financed spending increase and a deficit-financed tax cut. We follow the Mountford and Uhlig (2009) approach to construct policy experiments by linearly combining pure government spending and tax shocks. Our results show that for the sign-restrictions approach, the sign of the fiscal multiplier crucially depends on whether increases in government spending are tax-financed or deficit-financed. In contrast, the results for the Blanchard-Perotti approach suggest that the way

Accounts. Perotti (2005) argues that the accrual measure is preferable "because the cash adjustment displays a marked seasonality that is difficult to eliminate". To test for the importance of the different tax measures, we re-estimate the three-equation VAR used in Blanchard and Perotti (2002) using the Perotti (2005) tax measure. The results of this exercise suggest that the differences in the output response across these two studies are largely attributable to the different tax measures. Detailed results are available upon request.
2.2. DATA

in which government spending is financed is of no importance, which is in line with the assumption of Ricardian Equivalence commonly made in recent theoretical literature. In our view, the uncertainty about whether Ricardian Equivalence is a good approximation of economic reality again points to the importance of a better modeling and understanding of the effects of tax shocks.

The remainder of this paper is organized as follows. Section 2.2 describes the data used for our comparative analysis. Section 2.3 presents the econometric methodology, including a description of the reduced-form VAR model and the alternative identification approaches. Section 2.4 presents the results for pure government spending and tax shocks. Section 2.5 analyzes the relationship between the estimated size of automatic stabilizers and the estimated output effects of exogenous tax shocks. Section 2.6 presents results for the policy experiments. Section 2.7 presents the results of a sensitivity analysis and Section 2.8 concludes the paper.

2.2 Data

We use quarterly U.S. data over the period 1955:1 – 2006:4. The components of national income and various fiscal series are drawn from the NIPA tables published by the Bureau of Economic Analysis. The interest rate series is drawn from the Federal Reserve Bank of Saint Louis’ ALFRED database. Our baseline measure of the real wage (real hourly compensation in the business sector) is drawn from the Bureau of Labor Statistics, while our baseline measure of employment (total economy hours worked per capita) is taken from Francis and Ramey (2005). The data appendix gives details on definitions and data sources for all variables used in the baseline and sensitivity analyses.

Our baseline model is a five-variable VAR model including the log of real per capita government spending, \( g_t \), the log of real per capita net taxes, \( \tau_t \),
the log of real per capita GDP, $y_t$, the GDP deflator inflation rate, $\pi_t$, and a short-term interest rate, $r_t$. This set of variables is the same as that used by Perotti (2005). In addition, we specify six-variable VAR models, adding in turn the log of real per capita private consumption, $c_t$, the log of real per capita private nonresidential investment, $i_t^{NR}$, the log of real per capita private residential investment, $i_t^R$, the log of per capita hours worked, $n_t$, and the log of the real wage, $w_t$, to the set of variables.

Our definition of the fiscal variables closely follows the related literature. In particular, government spending and taxes are defined net of social transfers. More specifically, government spending is the sum of government consumption and investment, while net taxes are defined as government current receipts less current transfer and interest payments. Figure 1 shows the evolution of the government spending to GDP ratio and the net tax to GDP ratio over the period 1955-2006. The figure reveals some well-known fiscal episodes. As regards the spending ratio, one can discern the increase in the mid-1960s at the onset of the Vietnam war, the increase around 1980 associated with the Carter-Reagan military build-up, the drop in the 1990s associated with expenditure restraint under the Budget Act of 1990 and the Balanced Budget Act of 1997 and, more recently, the renewed increase related to military spending in the context of the war on terrorism following 9/11. As regards the tax ratio, the figure reveals the strong drops in the mid-1970s, the early 1980s and early 2000s, all related to both discretionary tax cuts and economic downswings, but also the strong increase during the stock-market boom in the late 1990s.

As one of the aims of this study is to provide evidence for those variables which are helpful to discriminate among competing theories, we also check whether our baseline results are robust to alternative variable definitions. As regards private consumption, we provide evidence for its durable and non-durable subcomponents. Our baseline measure of employment (total economy hours worked) includes hours worked in the government sector in order to
account for the fact that government wages constitute a large fraction of government consumption (see Cavallo, 2005). We also provide evidence for three alternative measures of employment: hours worked in the private business sector as well as the number of individuals employed in the private business sector and the government sector. Our baseline measure of wages (real hourly compensation in the business sector) is a measure of the real product wage relevant for firms’ hiring decisions. We also provide evidence for an alternative definition of the product wage as well as for two measures of the consumption wage relevant for households’ labor-supply decisions. Section 2.7.3 presents the results.

2.3 Econometric Methodology

This section presents the vector autoregressive methodology used in the empirical application. It first presents the benchmark reduced-form VAR model and then discusses how we implement the various identification approaches. Collecting the endogenous variables in the $k$-dimensional vector $X_t$, the reduced-form VAR model can be expressed as

$$X_t = \mu_0 + \mu_1 t + A(L)X_{t-1} + u_t,$$

(2.1)

where $\mu_0$ is a constant, $t$ is a linear time trend, $A(L)$ is a fourth-order lag polynomial and $u_t$ is a $k$-dimensional vector of reduced-form disturbances with $E[u_t] = 0$, $E[u_t u'_s] = \Sigma_u$ and $E[u_t u'_s] = 0$ for $s \neq t$. We follow Blanchard and Perotti (2002) and choose a lag length of four quarters. This seems to be a natural choice in a model with quarterly data and, moreover, using a higher lag order like, e.g., Mountford and Uhlig (2009) does not affect the results. Deterministic terms other than the constant and the linear time trend like the quadratic time trend, the seasonal dummy variables and the quarter-dependent coefficients considered by Blanchard and Perotti (2002) turned
out to be insignificant, thus they have been dropped.\textsuperscript{5} In our implementation of the event-study approach, we augment our baseline VAR model with a dummy variable capturing the onset of the Vietnam war in 1965, the Carter-Reagan military buildup in 1980 and the Iraq War in 2001.

We follow Mountford and Uhlig (2009) and estimate the VAR model using Bayesian methods. The main advantage of the Bayesian approach is that it allows for a conceptually clean way of drawing error bands for impulse responses (see Sims and Zha, 1999).\textsuperscript{6} We use a Normal-Wishart prior for the coefficient matrices $A(L)$ and $\Sigma_u$, implying that the posterior also belongs to the Normal-Wishart family. We take 500 draws from the posterior of the reduced-form VAR model and, for each draw of the posterior, identify the structural shocks for the three identification approaches discussed below. In Sections 2.4-2.7, we provide results in terms of impulse responses, reporting the median of the posterior distribution of the responses as well as error bands based on the 16\% and 84\% fractiles of the posterior distribution.\textsuperscript{7}

As the reduced-form disturbances will in general be correlated, it is necessary to transform the reduced-form model into a structural model. Pre-multiplying the above equation by the $(k \times k)$ matrix $A_0$ gives the structural form

$$A_0X_t = A_0\mu_0 + A_0\mu_1t + A_0A(L)X_{t-1} + B\epsilon_t,$$

\textsuperscript{5}Mountford and Uhlig (2009) do not include any deterministic terms in their reduced-form VAR model. Uhlig (2005) argues that this may result in a slight misspecification, but makes for more robust results because of the interdependencies in the specification of the prior between these terms and the roots in the autoregressive coefficients. In order to test whether our results are robust to the exclusion of deterministic terms, we also estimate our VAR models excluding the constant and the linear trend. The results are not affected qualitatively and there are only minor quantitative differences at longer horizons, with fiscal shocks exhibiting somewhat stronger long-run effects.

\textsuperscript{6}The main conclusions are not affected by the choice of a Bayesian approach rather than a classical approach. As regards the empirical results presented in this paper, the median impulse responses obtained using the Bayesian approach are nearly identical to the point estimate of the responses obtained using the classical approach.

\textsuperscript{7}See Uhlig (2005) for technical details on the estimation approach.
where $Be_t = A_0 u_t$ describes the relation between the structural disturbances $e_t$ and the reduced-form disturbances $u_t$. In the following, it is assumed that the structural disturbances $e_t$ are uncorrelated with each other, i.e., the variance-covariance matrix of the structural disturbances $\Sigma_e$ is diagonal. The matrix $A_0$ describes the contemporaneous relation among the variables collected in the vector $X_t$. In the literature, this representation of the structural form is often called the $AB$ model (see e.g. Lütkepohl, 2005). Without restrictions on the parameters in $A_0$ and $B$, the structural model is not identified. In the following, we present the identification approaches used in the empirical application.

### 2.3.1 The Recursive Approach

The first approach we consider is the recursive approach which restricts $B$ to a $k$-dimensional identity matrix and $A_0$ to a lower triangular matrix with a unit diagonal, which implies the decomposition of the variance-covariance matrix $\Sigma_u = A_0^{-1} \Sigma_e (A_0^{-1})'$. This decomposition is obtained from the Cholesky decomposition $\Sigma_u = PP'$ by defining a diagonal matrix $D$ which has the same main diagonal as $P$ and by specifying $A_0^{-1} = PD^{-1}$ and $\Sigma_e = DD'$, i.e. the elements on the main diagonal of $D$ and $P$ are equal to the standard deviation of the respective structural shock. The recursive approach implies a causal ordering of the model variables. Note that there are $k!$ possible orderings in total. In this paper, we order the variables as follows: spending is ordered first, output is ordered second, inflation is ordered third, tax revenue is ordered fourth and the interest rate is ordered last. This implies that the relation between the reduced-form disturbances $u_t$ and the structural disturbances $e_t$ takes the following form:

\footnote{See e.g. Lütkepohl (2005).}
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-\alpha_{yg} & 1 & 0 & 0 & 0 \\
-\alpha_{\pi y} & -\alpha_{\pi y} & 1 & 0 & 0 \\
-\alpha_{\tau y} & -\alpha_{\tau y} & -\alpha_{\pi \tau} & 1 & 0 \\
-\alpha_{rg} & -\alpha_{rg} & -\alpha_{\pi \tau} & -\alpha_{\pi \tau} & 1
\end{bmatrix}
\begin{bmatrix}
u^g_t \\
v^\pi_t \\
u^\tau_t \\
u^r_t
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
e^g_t \\
e^\pi_t \\
e^\tau_t \\
e^r_t
\end{bmatrix} \quad (2.3)
\]

This particular ordering of the variables has the following implications: (i) Government spending does not react contemporaneously to shocks to other variables in the system, (ii) output does not react contemporaneously to tax, inflation and interest rate shocks, but is affected contemporaneously by spending shocks, (iii) inflation does not react contemporaneously to tax and interest rate shocks, but is affected contemporaneously by government spending shocks, (iv) taxes do not react contemporaneously to interest rate shocks, but are affected contemporaneously by government spending, output and inflation shocks, and (v) the interest rate is affected contemporaneously by all shocks in the system. Note that after the initial period, the variables in the system are allowed to interact freely, i.e., for example, tax shocks can affect output in all periods after the one in which the shock occurred.

The assumptions on the contemporaneous relations between the variables can be justified as follows: Movements in government spending, unlike movements in taxes, are largely unrelated to the business cycle. Therefore, it seems plausible to assume that government spending is not affected contemporaneously by shocks originating in the private sector. Ordering output and inflation before taxes can be justified on the grounds that shocks to these two variables have an immediate impact on the tax base and, thus, a contemporaneous effect on tax receipts. Thus, this particular ordering of variables captures the effects of automatic stabilizers on government revenue, while it rules out (potentially important) contemporaneous effects of discretionary tax changes on output and inflation. Ordering the interest rate last can be
2.3. ECONOMETRIC METHODOLOGY

justified (i) on the grounds of a central bank reaction function implying that
the interest rate is set as a function of the output gap and inflation, and (ii)
given that spending and revenue as defined here (net of interest payments)
are not sensitive to interest rate changes.

2.3.2 The Blanchard-Perotti Approach

The identification approach due to Blanchard and Perotti (2002) relies on
institutional information about tax and transfer systems and about the tim-
ing of tax collections in order to identify the automatic response of taxes
and government spending to economic activity. This identification scheme
relies on a two-step procedure: In a first step, the institutional information
is used to estimate cyclically adjusted taxes and government expenditures.
In a second step, estimates of fiscal policy shocks are obtained. Blanchard
and Perotti (2002) and Perotti (2005) applied this approach to estimate the
effects of government spending and tax shocks for the United States. This
subsection relies on the identification scheme used by Perotti (2005) as he
also used a five-variable VAR model while the Blanchard and Perotti (2002)
analysis built on a three-variable system. Adapting Perotti’s (2005) starting
point to our context, the relationship between reduced-form disturbances $u_t$
and structural disturbances $e_t$ can be written as

\begin{align*}
  u^g_t &= \alpha_{gy}u^y_t + \alpha_{g\pi}u^\pi_t + \alpha_{gr}u^r_t + \beta_{g\tau}e^\tau_t + e^g_t, \quad (2.4) \\
  u^\pi_t &= \alpha_{\pi y}u^y_t + \alpha_{\pi \pi}u^\pi_t + \alpha_{\pi r}u^r_t + \beta_{\pi g}e^g_t + e^\pi_t, \quad (2.5) \\
  u^y_t &= \alpha_{yy}u^g_t + \alpha_{yr}u^r_t + e^y_t, \quad (2.6) \\
  u^\tau_t &= \alpha_{\tau g}u^g_t + \alpha_{\tau y}u^y_t + \alpha_{\tau \pi}u^\pi_t + \alpha_{\tau r}u^r_t + e^\tau_t, \quad (2.7) \\
  u^r_t &= \alpha_{rg}u^g_t + \alpha_{ry}u^y_t + \alpha_{r\pi}u^\pi_t + e^r_t. \quad (2.8)
\end{align*}

Note that the above system of equations is not identified. The variance-
covariance matrix of the reduced-form disturbances has ten distinct elements
whereas the above system of equations has 17 free parameters. Unlike the recursive approach, the Blanchard-Perotti approach does not involve imposing (only) zero restrictions on seven parameters to achieve identification. The first step of the estimation strategy consists of an adjustment of government spending and revenue for the automatic response of these variables to the business cycle and inflation. For this purpose, Perotti (2005) regresses individual revenue items on their respective tax base, obtaining an aggregate value for the output elasticity of government revenue ($\alpha_{\tau y}$) of 1.85 and an aggregate value for the inflation elasticity of government revenue ($\alpha_{\tau \pi}$) of 1.25.

Since government spending is defined net of transfers and, thus, is acyclical, Perotti (2005) sets the output elasticity of government spending ($\alpha_{gy}$) equal to zero. He sets the inflation elasticity of government spending ($\alpha_{g\pi}$) equal to $-0.5$, arguing that nominal wages of government employees, which account for a large part of government consumption, do not react contemporaneously to changes in inflation, thus implying that the government wage bill declines in real terms if there is an unanticipated increase in inflation. In addition, he sets the interest rate elasticities of government spending ($\alpha_{gi}$) and net taxes ($\alpha_{\tau i}$) equal to zero, respectively, because interest payments paid and received by the government are excluded from the definition of spending and net taxes. Finally, he sets the parameter $\beta_{g\tau}$ equal to zero, which is equivalent to saying that government decisions on spending are taken before decisions on revenue. Imposing these restrictions on the parameter values, the relation between reduced-form and structural disturbances can be written in matrix form as:

\[ \begin{align*}
A_0 & = \begin{bmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{bmatrix} \\
B & = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}
\end{align*} \]

Since the structural parameters collected in $A_0$ and $B$ are nonlinearly related to the reduced-form parameters, a closed form of the maximum likelihood estimates does not exist, necessitating the use of an iterative optimizing algorithm to compute the estimates. We use the Broyden-Fletcher-Goldfarb-Shanno algorithm implemented in RATS Doan (2004).
2.3. ECONOMETRIC METHODOLOGY

Comparing this system of equations with the system for the recursive approach reveals the following differences between the two identification approaches: Whereas in the recursive approach all elements of $A_0$ above the principal diagonal are restricted to zero, there are three exceptions in Perotti’s identification approach. These exceptions are potentially important when the responses to a tax shock are considered. By fixing the size of automatic stabilizers, Perotti (2005) is able to freely estimate the contemporaneous effect of taxes on output and inflation whereas the recursive approach freely estimates the size of automatic stabilizers while imposing a zero restriction on the contemporaneous effect of taxes on output and inflation. Surprisingly, the empirical analysis suggests that the conceptual differences between the recursive approach and the Blanchard-Perotti approach have little effect on the results—for the benchmark value of the output elasticity of net taxes imposed for the Blanchard-Perotti approach.

2.3.3 The Sign-Restrictions Approach

The third approach identifies fiscal policy shocks via sign restrictions on the impulse responses. Unlike the recursive approach and the Blanchard-Perotti approach, the sign-restrictions approach does not require the number of shocks to be equal to the number of variables and it does not impose any linear restrictions on the contemporaneous relation between reduced-form and structural disturbances. Rather, Mountford and Uhlig (2009) impose
restrictions directly on the shape of the impulse responses and identify four shocks: a business cycle shock, a monetary policy shock, a government spending shock and a tax shock. In our application, we identify a business cycle shock, a government spending shock and a tax shock. We disregard the monetary policy shock because it is not the focus of this paper and because the results are not sensitive to the (non)identification of this shock. We impose the following sign restrictions on the impulse responses: The business cycle shock is identified by the requirement that the impulse responses of output and taxes are positive for at least the four quarters following the shock. This turns out to be the crucial identifying assumption, having implications also for the identification of the fiscal policy shocks. The tax shock is identified by the requirements that the impulse responses of taxes are positive for at least the four quarters following the shock, while the government spending shock is identified by the requirements that the impulse responses of government spending are positive for at least the four quarters following the shock. In addition, both shocks are required to be orthogonal to the business cycle shock identified in the first step. The assumption that the business cycle shock comes first rules out that the responses of the model variables to a fiscal policy shock all have the same sign as those to a business cycle shock. In practice, this assumption brings about that whenever taxes and output move in the same direction, this is attributed to a change in the business cycle. Thus, it is unlikely that an increase (fall) in taxes generates an increase (fall) in output, a phenomenon which has received some attention in the recent literature on the effects of fiscal policy under the label expansionary fiscal contractions (see e.g. Giavazzi et al. 2000). As a consequence, it might be that the sign-restrictions approach overstates the (negative) output effects of a tax shock.

Following Uhlig (2005), we write the relationship between the reduced-form disturbances $u_t$ and the structural shocks $e_t$ as $u_t = Be_t$, with $E[u_t u'_t] = \Sigma_u$ and $E[e_t e'_t] = I$. Note that $e_t$ is an $m$-dimensional vector with $m \leq k$, i.e.
2.3. ECONOMETRIC METHODOLOGY

unlike in the two approaches discussed above it is not necessary to identify as many shocks as there are variables. In our setup, for example, we identify three shocks using the sign-restrictions approach while there are five or six variables in the estimated VAR models. For the implementation of the sign-restrictions approach, Mountford and Uhlig (2009) decompose the matrix $B$ into two components, $B = PQ$, where $P$ is the lower triangular Cholesky factor of $\Sigma_u$ and $Q$ is an orthonormal matrix with $QQ' = I$. Note that the matrix $P$, which serves to identify the structural shocks in the recursive approach, here merely serves a useful computational tool without affecting the results. Instead, the matrix $Q$ plays the crucial role in the sign-restrictions approach because it collects the identifying weights with each column of $Q$ corresponding to a particular structural shock. We use the penalty function approach described in detail in Mountford and Uhlig (2009) to compute the individual elements of $Q$. The penalty function approach consists of minimizing a criterion function, which penalizes impulse responses violating the sign restrictions, with respect to the identifying weights. We take a number of draws from the posterior of the VAR coefficients and the variance-covariance matrix of the reduced-form residuals. For each draw, we identify three structural shocks. In all estimations we take as many draws as is necessary to obtain 500 draws satisfying the sign restrictions.

2.3.4 The Event-Study Approach

Following the work of Ramey and Shapiro (1998), parts of the literature have tried to avoid the identification problem inherent in structural VAR analysis and have instead looked for fiscal episodes which can be seen as exogenous with respect to the state of the economy. Ramey and Shapiro (1998) have argued that the large increases in military spending associated with the onset of the Korean war, the Vietnam war and the Reagan military buildup can be seen as such exogenous events. Later, Eichenbaum and Fisher (2005) have argued that the expansion of defense spending in the aftermath of 9/11 can
also be viewed as such an exogenous event. We follow the literature and define a dummy variable, $D_t$, which takes on the value of 1 in the first quarter of 1965, i.e. at the onset of the Vietnam war, in the first quarter of 1980, i.e. at the onset of the Reagan military buildup, and in the third quarter of 2001, i.e. at the onset of the war on terrorism following 9/11. Our sample excludes the Korean war, which occurred in the early 1950s.\footnote{This omission affects the results for the event-study approach. Perotti (2007) shows that the consumption response to a spending increase is negative if the Korean war is included in the analysis, while it is positive if it is excluded. We opted for the sample starting in 1955 for two reasons. First, it avoids our results being affected by the lagged effects of World War II. Second, Perotti (2007) shows that the military build-up associated with the Korean War was very different in nature from later episodes in that it was entirely tax-financed.} Including the dummy variable in the empirical model, our baseline reduced-form VAR model given by equation (1) is replaced by the following reduced form:

$$X_t = \mu_0 + \mu_1 t + A(L)X_{t-1} + \Phi (L) D_t + u_t,$$

where $\Phi (L)$ is the fourth-order lag polynomial associated with the dummy variable capturing the above-mentioned fiscal episodes.

### 2.4 Results for the Pure Fiscal Shocks

This section presents empirical results for pure government spending and tax shocks, i.e. for shocks to one fiscal variable at a time without constraining the response of the respective other fiscal variable. Instead, Section 2.6 presents results for selected policy experiments. The impulse responses presented in this section are scaled as follows: As regards the responses of output and its components as well as the fiscal variables, the original impulse responses are transformed such as to give the dollar response of each variable to a dollar shock in one of the fiscal variables.\footnote{For the event-study approach, the impulse responses are not transformed using this method because the impact change in government spending is close to zero for this ap-}
2.4. RESULTS FOR THE PURE FISCAL SHOCKS

dure of Blanchard and Perotti (2002) and first divide the original impulse responses by the standard deviation of the respective fiscal shock in order to have shocks of the size of one percent. These impulse responses are then divided by the ratio of the respective variable and the shocked fiscal variable, where the ratio is evaluated at the sample mean. The major advantage of this transformation is that the responses of output to the fiscal shocks can be interpreted as (non-accumulated) multipliers. As regards the responses of inflation, wages and employment, they give the percentage change of each variable in response to a one-percent fiscal shock. Finally, the responses of the interest rate are expressed as a change in percentage points for a one-percent fiscal shock. For each variable, we report the median as well as the 16% and 84% fractiles of the posterior distribution of the impulse responses.

2.4.1 The Pure Spending Shock

The impulse responses for a pure spending shock are shown in Figure 2, with the individual columns displaying the results for the alternative identification approaches.\textsuperscript{12} The figure reveals a number of interesting findings. First, the identified government spending shocks are the same for all identification approaches except the event-study approach.\textsuperscript{13} According to the latter approach, government spending does not change to any considerable extent at the onset of a fiscal episode whereas according to the other approaches the increase in government spending is close to its peak on impact. Second, the results show that taxes on impact at most partly offset the increase in government spending is close to its peak on impact. Instead, we report the percentage change in all variables in response to a unit increase in the dummy variable capturing the Ramey-Shapiro episodes.

\textsuperscript{12}In all figures we use the following acronyms: RA for the recursive approach, BP for the Blanchard-Perotti approach, SR for the sign-restrictions approach and ES for the event-study approach. The acronyms used for the variables are explained in the Data Appendix.

\textsuperscript{13}In the case of the recursive approach and the Blanchard-Perotti approach, not only the responses of government spending but also all other responses are virtually identical. This is not surprising given that the spending shock is identified in the same way for both approaches, namely by ordering government spending first (compare the first row of matrix $A$ in equations (2.3) and (2.9)).
crease in government spending. For the sign-restrictions approach, e.g., taxes hardly change at all in the first year suggesting that in this case, the pure spending shock can be interpreted as a deficit-financed spending shock. In contrast, taxes increase by about 50 cents on impact for the recursive and the Blanchard-Perotti approaches. These differences in the tax responses potentially limit the comparability of the results across approaches. We tackle this issue in Section 2.6. Third, despite the differences in tax responses for alternative identification approaches, the responses of non-fiscal variables show striking similarities. For all approaches, real GDP persistently increases in response to a government spending shock, following a hump-shaped pattern. Moreover, for all approaches but the event-study approach, the spending multiplier peaks after three to four years at a value of around 2. A persistent and hump-shaped increase is also obtained for the response of private consumption.\textsuperscript{14} The results also show that the hours worked do not change significantly for all approaches considered, while the real product wage strongly and persistently increases according to all approaches but the event-study approach. According to the latter approach, the real wage falls somewhat but the responses are not statistically significant at any horizon. As regards private residential and nonresidential investment, the responses are small in general and not statistically significant. Inflation and the short-term interest rate increase with a lag of around two years.

As discussed in the introduction, one branch of the theoretical literature has taken as a stylized fact that private consumption increases in response to a government spending shock. The evidence presented here suggests that private consumption indeed increases in response to a positive government spending shock and that the responses of labor market variables seem to be important to rationalize the consumption response. Our empirical results

\textsuperscript{14}In the case of the sign-restrictions approach, the output and consumption responses are not statistically significant at horizons up to one year. In the case of the event-study approach, the consumption and output responses die out more quickly than for the remaining approaches and are statistically significant only at the one to three year horizons.
support models which generate an increase in the real wage but, at the same time, do not support the increase in employment implied by most current-generation DSGE models. A further challenge arising from the empirical evidence is that the positive responses of private consumption and the real wage are very persistent, whereas most current-generation DSGE models consistent with an increase in these variables predict that the responses turn negative already about one year after the government spending shock occurs (see e.g. Gali et al., 2007).

2.4.2 The Pure Tax Shock

The impulse responses for a pure tax shock are shown in Figure 3. Results are presented for all approaches but the event-study approach, which is only suitable for the analysis of spending shocks. The results shown in the figure reveal that while all approaches agree on the responses of fiscal variables, there is a strong discrepancy as regards the responses of non-fiscal variables. Turning first to the fiscal variables, the tax response peaks in the quarter when the shock occurs and then monotonically declines to die out after about three years while government spending does not react at all. The pure tax shock can, thus, be interpreted as a deficit-reducing tax increase policy experiment. As regards the responses of the non-fiscal variables, there is some disagreement on the effects of pure tax shocks between the recursive and Blanchard-Perotti approaches on the one hand and the sign-restrictions approach on the other hand. While the results for the latter approach suggest that unanticipated tax increases have strong distortionary effects, the results for the two other approaches suggest that tax shocks hardly have any effects on the real economy. In the case of the sign-restrictions the decline

\textsuperscript{15}Our results regarding the consumption, employment and wage responses to government spending shocks confirm the evidence presented in Perotti (2007).

\textsuperscript{16}In the case of the recursive and Blanchard-Perotti approaches, only the labor market variables show statistically significant responses, with hours worked declining and the real product wage increasing in response to a tax shock. The increase in the product wage
in GDP peaks at about 1.2 dollars after around one year whereas the GDP response is never significantly different from zero according to the other two approaches.

A surprising finding is that the results for the recursive approach and the Blanchard-Perotti approach are nearly identical also for the pure tax shock. A priori, it could be expected that the results for these approaches differ because the recursive approach restricts the short-run output effect of a pure tax shock to be zero while the Blanchard-Perotti approach does not. Our baseline results suggest that this conceptual difference is of no great importance given that the impact response of output is close to zero for the Blanchard-Perotti approach. This result is in line with the results reported by Perotti (2005) but stands in contrast to the results of Blanchard and Perotti (2002) who report that output decreases by around 70 cents on impact in response to a pure revenue shock.\footnote{Perotti (2005) presents empirical evidence for five OECD countries. The impact output response to an unanticipated tax increase is zero in four cases and even positive in the case of Australia (note that Perotti presents results for a tax cut). For two countries (Australia and the United Kingdom) the output response is positive at horizons larger than 1.} In contrast, our results for the sign-restrictions approach are similar in magnitude to those reported by Blanchard and Perotti (2002). Before turning to the implications of the tax shock results for selected policy experiments (see Section 2.6), the next section provides evidence explaining the differences in results obtained for the pure tax shock.

\section{The Size of Automatic Stabilizers}

This section shows that the striking coincidence of results for the recursive and Blanchard-Perotti approaches as well as the strong disagreement between these two approaches and the sign-restrictions approach can be traced back to the same underlying source: the size of automatic stabilizers estimated can be attributed to the fact that this is a gross wage including labor taxes and social contributions.
2.5. THE SIZE OF AUTOMATIC STABILIZERS

or calibrated for alternative identification approaches. At first sight, it may seem surprising that the size of automatic stabilizers is of importance for the size and even the sign of the macroeconomic effects of discretionary tax changes. Yet, there is a simple reason why they must be closely related: What the identification of tax shocks does is to separate the correlation between the residuals in the GDP equation and the residuals in the tax equation into two components—the automatic response of taxes to unexpected changes in GDP (automatic stabilizers) and the response of GDP to unexpected changes in taxes not related to the business cycle (the output effects of discretionary tax changes). For example, the estimation results for the reduced-form five-equation VAR model show that the correlation coefficient between the tax residuals and the GDP residuals is positive and equal to 0.42.18 This positive correlation could a priori be compatible with very different views about the relation between taxes and output: first, it could be exclusively due to automatic stabilizers, implying that discretionary tax shocks do not have any effects (this is suggested by the results for the baseline recursive and Blanchard-Perotti approaches); second, it could be the case that automatic stabilizers taken on their own would suggest an even larger positive correlation, partly offset by negative effects of discretionary tax increases (this is what is suggested by the results for the sign-restrictions approach); and, third, it could be that automatic stabilizers are not sufficiently large to explain all of the positive correlation but that the remainder is explained by discretionary tax changes not having negative but instead positive output effects in line with the literature on “expansionary fiscal contractions”.

Figure 4 presents evidence underlining the empirical relevance of the explanations given above. The figure shows the impact response of GDP to a pure revenue shock, i.e. the GDP response in the period when the tax shock occurs, for alternative values of the output elasticity of net taxes ($\alpha_{\tau y}$) using

---

18Blanchard and Perotti (2002) report a very similar value (0.38) for their three-variable VAR model estimated over the period 1960-1997.
the Blanchard-Perotti approach. Recall that $\alpha_{\tau y}$ measures the size of automatic stabilizers in the Blanchard-Perotti approach. For the baseline results presented in Section 2.4.2 we calibrate the output elasticity of net taxes using the same value as Perotti (2005), i.e. we set $\alpha_{\tau y}$ equal to 1.85. For this value, the impact response of GDP to a pure revenue shock is close to zero. The figure also shows the impact response of GDP for other calibrated values of $\alpha_{\tau y}$ over the range from 0 to 4. As can be seen, the relationship between the output elasticity of net taxes and the impact response of GDP is almost linear. In the absence of automatic stabilizers ($\alpha_{\tau y} = 0$) the impact response of GDP is positive and amounts to around 65 cents. The impact response remains positive but becomes successively smaller as the output elasticity of net taxes increases until the latter reaches a value around 1.9. The impact response of GDP becomes negative as the output elasticity exceeds 2. These results show how sensitive the results for the Blanchard-Perotti approach are to the calibrated value of the output elasticity of net taxes.

Now it is also easy to understand why the baseline results for the recursive approach and the Blanchard-Perotti approach are nearly identical. Recall that in the recursive approach, the output elasticity of net taxes is treated as a free parameter in the estimation. We obtain a point estimate of 1.93, which is nearly identical to the value imposed for the Blanchard-Perotti approach. This explains the striking similarity of the results for these two approaches. Moreover, it is easily seen why the recursive approach is not well-suited for the analysis of tax shocks. By ordering the variables, the recursive approach either sets the impact response of GDP equal to zero (GDP ordered before taxes) or sets the size of automatic stabilizers equal to zero (taxes ordered before GDP). It is because we have chosen the first of these two options as our baseline that the results for the recursive and Blanchard-Perotti approaches are so strikingly similar.

The figure also shows that it would be possible to obtain results for the Blanchard-Perotti approach which resemble those obtained for the sign-
restrictions approach. Recall that Blanchard and Perotti (2002) and Perotti (2005) calibrate the value of $\alpha_{\tau y}$ based on a first-step cyclical-adjustment procedure applied outside the VAR model. Instead, it is possible to estimate rather than impose the output elasticity of net taxes inside the structural VAR model for the Blanchard-Perotti approach. For this purpose, it is necessary to restrict another parameter in the estimation and we choose to set $\beta_{\tau g}$ equal to zero in order to ensure identification.\footnote{Setting this parameter to zero does not affect the results. This statement is based on the benchmark Blanchard-Perotti structural VAR model where setting this parameter to zero gives an overidentifying restriction which can be tested and cannot be rejected in our context.} If the output elasticity of net taxes is freely estimated, a point estimate of 2.98 is obtained, which translates into a negative impact response of GDP to a revenue shock of almost 50 cents.\footnote{The standard deviation of the estimate of $\alpha_{\tau y}$ is 1.01, revealing that the uncertainty surrounding this parameter is quite large.} The results provided by the Blanchard-Perotti approach would, thus, appear not to differ significantly from those obtained for the sign-restrictions approach for the pure tax shock and, consequently, for the policy experiments. Moreover, the results for the sign-restrictions approach suggest that a value for the output elasticity of net taxes of 2.98 is not implausibly large. Figure 5 shows the responses of GDP and government revenue for the business cycle shock scaled such that the impact response of GDP is equal to 1 percent. The impact response of government revenue to the business cycle shock can be interpreted as measuring the size of automatic stabilizers and is, thus, comparable to the output elasticity of net taxes in the Blanchard-Perotti approach. The results suggest that government revenue increases by 3.49 percent on impact in response to the business cycle shock, with a 68% confidence band ranging from 2.97 to 3.90.

All in all, the results provided in this section suggest that there is considerable uncertainty regarding the size of automatic stabilizers. The recursive approach and the "calibrated" version of the Blanchard-Perotti approach suggest that automatic stabilizers are relatively small compared to the sign-
restrictions approach and the "estimated" version of the Blanchard-Perotti approach. The uncertainty about the magnitude of automatic stabilizers translates into uncertainty about the degree of distortion associated with a given tax shock and, as is shown in Section 2.6 below, also about the effects of policy experiments. We interpret our results as indicating a need for a refinement of the way taxes are adjusted for the effects of the business cycle in structural VAR models.

2.6 Results for the Policy Experiments

Most studies in the literature report the effects of pure fiscal shocks only. However, as pointed out by Mountford and Uhlig (2009), pure fiscal shocks are not connected to the policy experiments considered in the theoretical literature or by policymakers. The reason is that pure fiscal shocks do not restrict the time paths of both fiscal variables. As a consequence, it is not possible to answer such questions as “What are the effects of a tax-financed compared to a deficit-financed spending increase?” on the basis of the results for the pure spending shock because the identification of this shock does not restrict the response of taxes. Yet, there is an easy way of constructing meaningful policy experiments on basis of the results for pure spending and tax shocks. Following Mountford and Uhlig (2009), such policy experiments can be constructed as linear combinations of the two pure fiscal shocks. This section presents the results for three alternative policy experiments: a deficit-financed spending increase, a balanced-budget spending increase and a deficit-financed tax cut.

2.6.1 The Deficit-Financed Spending Increase

The deficit-financed spending increase is defined as an increase in government spending by 1$ for four quarters while taxes remain unchanged. This policy experiment is obtained by linearly combining the sequence of the two pure
fiscal policy shocks that cause these responses in the two fiscal variables. The impulse responses for this policy experiment are shown in Figure 6. As can be expected from the discussion in Section 4.1, the dynamics of the non-fiscal variables are very similar to those reported for the pure government spending shock. As regards the recursive and the Blanchard-Perotti approaches, this similarity stems from the results obtained for the pure tax shock indicating that tax shocks do not have any significant effects on non-fiscal variables. For the sign-restrictions approach, this similarity is due to the fact that taxes do not react to pure government spending shocks, thus implying that pure spending shocks are deficit-financed. The main message from Figure 6 is that output, private consumption and the real product wage all increase in response to a deficit-financed spending shock, while employment remains unchanged. Except for the output response these results are inconsistent with the standard neoclassical model. Yet, except for the employment response and the persistence of the other responses, they are consistent with the recent theoretical literature discussed in the Introduction.

2.6.2 The Balanced-Budget Spending Increase

The balanced-budget spending increase is defined as an increase in both government spending and taxes by \(1\) for four quarters. The impulse responses for this policy experiment are shown in Figure 7. As regards the recursive approach and the Blanchard-Perotti approach, the results for the balanced-budget spending increase are once more very similar to those reported in Figure 2 for the pure spending shock. For these approaches, it makes little difference whether a spending increase is deficit-financed or tax-financed because, as shown in Section 2.4.2, tax shocks hardly have any effect on the non-fiscal variables. In contrast, as regards the sign-restrictions approach, the results for the balanced-budget spending increase are markedly different from those reported for the pure spending shock (Figure 2) and for the deficit-financed spending increase (Figure 6). Output, consumption, investment and
hours worked all significantly fall for about two to three years in response to a balanced-budget spending increase. One possible interpretation of this finding is that the rise in distortionary taxes necessary to match the spending increase has strong disincentive effects which entail a decline in output. For example, the standard neoclassical growth model analyzed by Baxter and King (1993) predicts that output and employment decrease if a spending increase is financed with distortionary taxes while they increase if the increase is financed with lump-sum taxes (which is equivalent to deficit-finance in their model). All in all, the results for this policy experiment suggest that if one trusts the results for the recursive and Blanchard-Perotti approaches, the theoretical literature’s modeling choice of lump-sum taxes over distortionary taxes is innocuous whereas if one trusts the results for the sign-restrictions approach, this choice is problematic. Most importantly, according to the results for the latter approach, the sign of the fiscal multiplier depends on the financing alternative.

2.6.3 The Deficit-Financed Tax Cut

The deficit-financed tax cut is defined as a fall in taxes by $1 for four quarters while government spending remains unchanged. The impulse responses for this policy experiment are shown in Figure 8. The impulse responses are the mirror image of the responses depicted in Figures 3 for the pure tax shock. The main reason for this similarity is that government spending does not respond very strongly to a pure tax shock, implying that the pure tax shock can be interpreted as a deficit-reducing tax increase. Thus, for the deficit-financed tax cut, the results for the recursive and Blanchard-Perotti approach indicate that none of the non-fiscal variables shows any significant response, whereas the results for the sign-restrictions approach suggest that output, consumption, investment, employment, inflation and the interest rate increase in the short to medium run, while the real product wage falls. Put differently, the recursive and Blanchard-Perotti approaches suggest that Ri-
2.7. ROBUSTNESS

Cardian Equivalence is a good approximation of economic reality, while the sign-restrictions approach suggests that taxes are strongly distortionary. The uncertainty about whether Cardian Equivalence is supported by the data once more points to the importance of a better modeling and understanding of the effects of tax shocks.

2.7 Robustness

This section presents the results of various sensitivity analyses regarding the specification of the reduced-form VAR model, subsample stability and the use of alternative definitions for key variables.

2.7.1 Reduced-Form VAR Specification

As concerns the specification of the reduced-form VAR model, the results presented for the benchmark specification are robust to the following alternative specifications:²¹ (i) use of a sixth order lag polynomial instead of a fourth order lag polynomial, (ii) inclusion of a quadratic time trend among the deterministic terms, (iii) inclusion of a dummy variable capturing the Ramey and Shapiro (1998) episodes also in the baseline VAR model, (iv) inclusion of a dummy variable capturing the tax rebate in the second quarter of 1975, as in BP (2002), and (v) trend break in 1973:2 as in Burnside et al. (2004). The fact that different approaches agree under different specifications of the reduced-form VAR does not necessarily mean that specification issues are not relevant. We change one component of the reduced-form VAR at a time, while in the literature the specification of the reduced-form VAR models sometimes differs substantially. In particular, differences can arise because of the inclusion of a different number of variables, different definitions of the

²¹Only minor quantitative deviations from the baseline results are recorded. Detailed results are available upon request.
time series meant to capture the same concept, different data sources as well as different sample sizes and periods. For example, as mentioned earlier, the results for the event-study approach are sensitive to the starting date of the sample.

2.7.2 Subsample Stability

Perotti (2005) presents evidence suggesting that the transmission of fiscal policy shocks has changed over time. In particular, the responses of GDP and its components appear to have become weaker in the post-1980 period. Bilbiie et al. (2006) show that the same holds for the responses of the real wage. To check whether our main results are stable we split our sample into two subperiods: 1955:1-1979:3 (99 observations) and 1983:1-2006:4 (96 observations). We follow the literature and exclude the period 1979:4 to 1982:4 in order to avoid that the shorter samples are affected by the substantial changes in the monetary policy framework that occurred around that time.

Figure 9 reports the results for a pure spending shock for the two subsamples. The results suggest that for both subsamples, the pure spending shocks can be interpreted as deficit-financed spending increases. There is some evidence that the effects of spending shocks have become somewhat weaker and less persistent over time but the results do not suggest that the changes were dramatic. Most importantly, the qualitative effects of government spending shocks do not seem to have changed to any considerable extent over time. The responses of output, private consumption and the real product wage are significantly positive for both subsamples, with the response of consumption showing the familiar hump-shaped pattern. The only response that signifi-

\footnote{As an important example, consider the implications of the different definitions of net tax series used by Blanchard and Perotti (2002) and Perotti (2005) discussed in the Introduction.}

\footnote{Note that this choice of subsamples excludes the Carter-Reagan military build-up. For the event-study approach, the first subsample includes the Vietnam war while the second subsample includes the military operations following 9/11.}
2.7. ROBUSTNESS

cantly changes is the one of hours worked. The response is positive in the
first subsample but negative in the second subsample, which explains why
the full sample results suggest that there is no significant reaction in hours
worked. In any case, the current-generation DSGE literature is inconsistent
with either unchanged hours worked or declining hours worked.

2.7.3 Alternative Measures of Consumption, Employment and the Real Wage

Figure 10 presents impulse responses to a pure spending shock for alterna-
tive definitions of key variables using the baseline six-variable VAR model
estimated over the full sample. First, we split private consumption into its
durable and nondurable subcomponents. The results show that the consump-
tion of both durable and nondurable goods increases in response to a pure
spending shock, with the hump-shaped pattern being more pronounced in the
case of durable goods. Second, we show results for three alternative defini-
tions of employments. The baseline model uses total economy hours worked,
which do not react significantly to a spending shock. The same is true for
hours worked (the intensive margin) and the number of employees (the ex-
tensive margin) in the business sector. In contrast, the number of government
employees significantly increases in response to a spending shock, which is
in line with the empirical findings documented in the literature (see Cavallo,
2005). Finally, we show results for three alternative definitions of the real
wage. The baseline model uses the real product wage in the business sector,
which strongly increases in response to a spending shock. The same is true for
the real product wage in the manufacturing sector. In contrast, the increase
in real consumption wages in the business and manufacturing sectors is less
pronounced and in general statistically insignificant. All in all, we interpret
the evidence provided in this section as showing that the baseline findings
presented in this paper are quite robust.
2.8 Conclusions

This paper presents an extensive comparative study on the empirical literature using vector autoregressive models to assess the effects of fiscal policy shocks. The starting point of our analysis is that there is strong disagreement in the literature not only on the quantitative but also on the qualitative effects of fiscal policy shocks. We provide new evidence for the U.S. over the period 1955-2006. We show that, controlling for differences in the specification of the reduced-form model, all identification approaches used in the literature yield qualitatively and quantitatively very similar results as regards government spending shocks. In response to such shocks, real GDP, real private consumption and the real wage all significantly increase following a hump-shaped pattern, while there is no reaction in private employment.

Our empirical results support theoretical models which generate an increase in private consumption and the real wage but, at the same time, do not support the increase in employment implied by most current-generation DSGE models. A further challenge arising from the empirical evidence is that the positive responses of private consumption and the real wage are very persistent, whereas most current-generation DSGE models consistent with an increase in these variables predict that the responses turn negative already about one year after the government spending shock occurs.

In contrast, we find strongly diverging results as regards the effects of tax shocks depending on the identification approach used, with the estimated effects of unanticipated tax increases ranging from non-distortionary to strongly distortionary. We show that the differences in results can to a large extent be traced back to differences in the automatic response of tax revenues to the business cycle (automatic stabilizers) estimated or calibrated for alternative identification approaches, with the degree of distortion associated with a given tax shock being positively related to the estimated size of automatic stabilizers. This uncertainty about the effects of tax shocks also translates into uncertainty about the effects of policy experiments. As
2.8. CONCLUSIONS

regards the effects of balanced-budget spending increases, e.g., our results show that the sign of the fiscal multiplier depends on the degree to which taxes are estimated to be distortionary. We interpret our results as indicating a need for a better modeling of the effects of tax shocks and, in particular, for a refinement of the way in which taxes are adjusted for the effects of the business cycle in structural VAR models.

The latest studies in this literature have pointed out two interesting extensions to the baseline VAR models used to assess the effects of fiscal policy shocks. First, since fiscal policy measures are in general announced in advance of their implementation, the standard fiscal VAR model—assuming that fiscal policy shocks are unanticipated—might be misspecified. Two recent VAR-based studies accounting for announcement effects disagree on whether these effects are of importance. Tenhofen and Wolff (2007) provide evidence showing that the response of private consumption to a government spending shock turns negative once the empirical model accounts for announcement effects, whereas Mountford and Uhlig (2009) find that the response of private consumption is positive if announcement effects are accounted for. Yet, it is questionable whether VAR models are the appropriate tool for gauging the importance of announcement effects. Introducing fiscal policy foresight into an otherwise standard DSGE model, Yang (2005) shows that the data-generating process is not invertible and, as a consequence, does not have a VAR representation but instead a VARMA (vector autoregressive moving average) representation. Second, the standard VAR model does not explicitly take into account fiscal solvency considerations and thus, it cannot be ruled out a priori that the estimated fiscal shocks and their transmission imply explosive debt dynamics. Chung and Leeper (2007) address this issue by imposing a debt-stabilizing condition derived from the intertemporal government budget constraint on the estimated VAR model. Their results suggest that imposing fiscal solvency has quantitatively important implications at very long horizons, whereas responses at the horizons considered here
(up to ten years) are not strongly affected. Similarly, Favero and Giavazzi (2007) show that including government debt in the set of observable variables has important implications for the response of interest rates to fiscal policy shocks, whereas the responses of other macroeconomic variables—which are the focus of our analysis—are not strongly affected by the inclusion of this variable. We leave the detailed exploration of these issues to future research.

Bibliography


Lütkepohl, H., New introduction to multiple time series analysis (Springer, 2005).


Uhlig, H., “What are the effects of monetary policy on output? Results from an agnostic identification procedure,” *Journal of Monetary Economics* 52 (March 2005), 381–419.

Yang, S.-C., “Quantifying tax effects under policy foresight,” *Journal of Monetary Economics* 52 (November 2005), 1557–1568.

## A Data Appendix

The data were taken from four sources. The components of national income, government receipts and the GDP deflator were taken from the NIPA tables of the Bureau of Economic Analysis (vintage date: April 27, 2007). Of the remaining series, the interest rate series were taken from the ALFRED database of the Federal Reserve Bank of Saint Louis (vintage dates: May 8, 2007) while the labor market variables—with one exception—were taken from the Bureau of Labor Statistics (BLS) (download date: May 17, 2007). The Francis and Ramey (2005) measure of total hours worked per capita was downloaded from Ramey’s homepage http://econ.ucsd.edu/~vramey/ on 16 May 2007. The other series can be obtained free of charge from http://www.bea.gov/histdata/NIyear.asp, http://alfred.stlouisfed.org/ and http://www.bls.gov/, respectively. The components of national income and net taxes are in real per capita terms and were transformed from their nominal values by dividing them by the price index for GDP (NIPA Table 1.1.4, Line 1) and by civilian noninstitutional population (ALFRED Series ID: CNP16OV). All series are seasonally adjusted by the source. For all series except the interest rates, we took the natural logarithm and multiplied the resulting series by 100, thus yielding the series used in the estimation. Where necessary we take the arithmetic average of monthly figures to obtain quarterly series.

- **GDP**: ’Gross domestic product’; NIPA Table 1.1.5, Line 1.
• **Private consumption** (**P\_CONS**): 'Personal consumption expenditures'; NIPA Table 1.1.5, Line 2. 'Durable goods'; NIPA Table 1.1.5, Line 3. 'Nondurable goods'; NIPA Table 1.1.5, Line 4.

• **Government spending** (**G\_SPEN**): 'Government consumption expenditures and gross investment'; NIPA Table 1.1.5, Line 20).

• **Net taxes** (**TAX**): ‘Government current receipts’ (NIPA Table 3.1 Line 1) minus ‘Current transfer payments’ (NIPA Table 3.1 Line 17) minus ‘Government interest payments’ (NIPA Table 3.1, Line 22).

• **Residential investment** (**R\_INV**): ‘Private Fixed Investment - Residential’; NIPA Table 1.1.5, Line 11.

• **Nonresidential investment** (**NR\_INV**): ‘Gross Private Domestic Investment’ (NIPA Table 1.1.5, Line 6) minus ‘Private Fixed Investment - Residential’ (NIPA Table 1.1.5, Line 11).

• **Inflation** (**INFL**): Log difference of the price index for GDP (NIPA Table 1.1.4, Line 1).

• **Interest rate** (**INT**): '3-Month Treasury Bill: Secondary Market Rate’ (ALFRED Series ID: TB3MS).

• **Hours worked** (**HOURS**): 'Total Economy Weekly Hours per Capita’ from Francis and Ramey (2005).

• **Real compensation** (**W**): 'Real Hourly Compensation, Business Sector, Index 1992=100’ (BLS Series ID: PRS84006153), deflated by the source using the Consumer Price Index for all urban consumers (CPI-U).

The following series were used for the sensitivity analyses reported in subsection 2.7.3:
A. DATA APPENDIX

- **Long-term interest rate**: '10-Year Treasury Constant Maturity Rate' (ALFRED Series ID: GS10).

- **Hours worked - Business**: 'Private Business Sector Weekly Hours' divided by noninstitutional population aged 16+, both taken from Francis and Ramey (2005).

- **Employment - Business**: 'Total Private Employees' (BLS Series ID: CES0500000001).

- **Employment - Government**: 'Government Employees' (ALFRED Series ID: USGOVT).


- **Real wage - Business**: 'Average Hourly Earnings of Production Workers, Private Sector' (BLS Series ID: CES3000000008) divided by the CPI (BLS Series ID: CUSR0000SA0).

- **Real wage - Manufacturing**: 'Average Hourly Earnings of Production Workers, Manufacturing Sector' (BLS Series ID: CES3000000008) divided by the CPI (BLS Series ID: CUSR0000SA0).
B Figures

Figure 1: Net Taxes and Spending, Share of GDP

The solid line plots the ratio of government spending to GDP, the dotted line the ratio of net taxes to GDP over the period 1955-2006.
Figure 2: Responses to a Pure Spending Shock

The solid lines plot the median, the dotted lines the 16% and 84% fractiles of the posterior distribution of the impulse responses for the recursive identification approach (column 1), the Blanchard-Perotti approach (column 2), the sign-restrictions approach (column 3) and the event-study approach (column 4), for 40 quarters. For the first three approaches the responses of GDP, its components and the fiscal variables are scaled such that they depict the dollar change in these variables in response to a pure government spending shock of the size of one dollar. For the event study approach, the responses depict the percentage change of these variables to a pure government spending shock of the size of one percent. For inflation, hours worked, and real hourly compensation, the responses are scaled such that they depict the percentage change in response to a pure government spending shock of the size of one percent. For the real interest rate, the responses are scaled such that they depict the change in percentage points in response to a pure government spending shock of the size of one percent.
Figure 3: Responses to a Pure Tax Shock

The solid lines plot the median, the dotted lines the 16% and 84% fractiles of the posterior distribution of the impulse responses for the recursive identification approach (column 1), the Blanchard-Perotti approach (column 2) and the sign-restrictions approach (column 3), for 40 quarters. The responses of GDP, its components and the fiscal variables are scaled such that they depict the dollar change in these variables in response to a pure government spending shock of the size of one dollar. For inflation, hours worked, and real hourly compensation, the responses are scaled such that they depict the percentage change in response to a pure government spending shock of the size of one percent. For the real interest rate, the responses are scaled such that they depict the change in percentage points in response to a pure government spending shock of the size of one percent.
Figure 4: Impact Response of GDP to a Pure Tax Shock – BP Approach

The symbols depict the impact response of GDP in US dollars to a pure revenue shock of the size of one dollar for the Blanchard-Perotti identification approach for alternative values of the output elasticity of net taxes. The vertical lines indicate the values of the output elasticity of net taxes imposed by Perotti (2006), Blanchard and Perotti (2002) and the value that is obtained if this elasticity is treated as a free parameter in the estimation, respectively. The responses are based on the five-variable VAR model.
Figure 5: Responses to a Business Cycle Shock – Sign-Restrictions Approach

The solid lines plot the median, the dotted lines the 16% and 84% fractiles of the posterior distribution of the impulse responses for the sign-restrictions identification approach. The responses are shown for a horizon of 40 quarters. They depict the percentage change in the plotted variables in response to a business cycle shock standardized such that the impact response of GDP is equal to one percent. The sign restrictions on the impulse responses are indicated by the shaded areas. The responses are based on the five-variable VAR model.
Figure 6: Responses to a Deficit-Financed Spending Increase

The solid lines plot the median, the dotted lines the 16% and 84% fractiles of the posterior distribution of the impulse responses for the recursive identification approach (column 1), the Blanchard-Perotti approach (column 2) and the sign-restrictions approach (column 3), for 40 quarters. The pure spending and revenue shocks are linearly combined such that the response of government spending is equal to one dollar for four quarters and the responses of net taxes are equal to zero for four quarters. The responses of GDP, its components and the fiscal variables are scaled such that they depict the dollar change in these variables in response to a pure government spending shock of the size of one dollar. For inflation, hours worked, and real hourly compensation, the responses are scaled such that they depict the percentage change in response to a pure government spending shock of the size of one percent. For the real interest rate the responses are scaled such that they depict the change in percentage points in response to a pure government spending shock of the size of one percent.
Figure 7: Responses to a Balanced-Budget Spending Increase

The solid lines plot the median, the dotted lines the 16% and 84% fractiles of the posterior distribution of the impulse responses for the recursive identification approach (column 1), the Blanchard-Perotti approach (column 2) and the sign-restrictions approach (column 3), for 40 quarters. The pure spending and revenue shocks are linearly combined such that the responses of both government spending and government revenue are equal to one dollar for four quarters. The responses of GDP, its components and the fiscal variables are scaled such that they depict the dollar change in these variables in response to a pure government spending shock of the size of one dollar. For inflation, hours worked, and real hourly compensation, the responses are scaled such that they depict the percentage change in response to a pure government spending shock of the size of one percent. For the real interest rate, the responses are scaled such that they depict the change in percentage points in response to a pure government spending shock of the size of one percent.
Figure 8: Responses to a Deficit-Financed Tax Cut

The solid lines plot the median, the dotted lines the 16% and 84% fractiles of the posterior distribution of the impulse responses for the recursive identification approach (column 1), the Blanchard-Perotti approach (column 2) and the sign-restrictions approach (column 3), for 40 quarters. This policy experiment is defined as follows: The pure spending and revenue shocks are linearly combined such that the response of net taxes is equal to minus one dollar for four quarters while government spending remains unchanged. The responses of GDP, its components and the fiscal variables are scaled such that they depict the dollar change in these variables in response to a pure government spending shock of the size of one dollar. For inflation, hours worked, and real hourly compensation, the responses are scaled such that they depict the percentage change in response to a pure government spending shock of the size of one percent. For the real interest rate, the responses are scaled such that they depict the change in percentage points in response to a pure government spending shock of the size of one percent.
Figure 9: Responses to a Pure Spending Shock – Subsamples

See notes to Figure 2.
Figure 10: Responses to a Pure Spending Shock — Alternative Measures of Consumption, Employment and the Real Wage

The solid lines plot the median, the dotted lines the 16% and 84% fractiles of the posterior distribution of the impulse responses for the recursive identification approach (column 1), the Blanchard-Perotti approach (column 2) and the sign restrictions approach (column 3), for 40 quarters. The responses of durable and nondurable consumption are scaled such that they depict the dollar change in these variables in response to a pure government spending shock of the size of one dollar. For the alternative measures of hours worked, employment and the real wage, the responses are scaled such that they depict the percentage change in response to a pure government spending shock of the size of one percent.
Chapter 3

The Analytics of SVARs: A Unified Framework to Measure Fiscal Multipliers

3.1 Introduction

Governments often use fiscal policy to stabilize economic fluctuations. For example, during the recent recession, the United States Congress approved the American Recovery and Reinvestment Act, which introduced increases in public spending and cuts in taxes by approximately 6% of GDP (CBO, 2010b). The rationale for such fiscal stimulus rests on the assumption that fiscal interventions do stabilize the economy. Yet, the size of fiscal multipliers, defined as the dollar response of output to an exogenous dollar spending increase or tax cut, is the subject of a long-standing debate in academia. As

*I am indebted to Jesús Fernández-Villaverde, John Hassler, Frank Schorfheide, and Torsten Persson for valuable advice. I am also very grateful to Christophe Kamps for substantial feedback and very helpful comments. I thank seminar participants at the Sverige Riksbank, ECB, Bank of England, IIES, Stockholm School of Economics, University of Pennsylvania, and the Fourth Oslo Workshop on Economic Policy. Of course, all remaining errors are my own.
Perotti (2007) observes in his survey of the literature: "... perfectly reasonable economists can and do disagree on the basic theoretical effects of fiscal policy and on the interpretation of existing empirical evidence".

The presence of competing economic theories has motivated a large body of empirical investigations that measure the size of these fiscal multipliers. An important share of the literature relies on structural vector autoregressions (SVARs). Prominent examples include Blanchard and Perotti (2002), Perotti (2005), and Mountford and Uhlig (2009). The appeal of SVARs is that they control for endogenous movements in fiscal policies by only imposing a minimal set of assumptions, known as identification schemes. Yet, despite their simple structure and the use of similar data, studies employing SVARs document fiscal multipliers that are spread over a broad range of values. So far, little effort has been devoted to understanding which assumptions in competing SVARs drive differences in results. The lack of robust evidence prevents the profession from providing clear guidance on important policy choices, such as the size and composition of fiscal interventions.

Motivated by this lacuna of knowledge, my paper asks two questions. Why do SVARs provide different measures of fiscal multipliers? Can we construct robust measures of fiscal multipliers using SVARs?

I answer the first question by deriving a unified analytical framework to compare competing identification schemes. Then, I apply this analysis to a fiscal VAR for the United States for the period 1947-2010. I show that existing identification schemes imply different restrictions on the output elasticity of tax revenue and government spending. These elasticities measure the endogenous response of tax and spending policies to economic activity. For instance, the Blanchard and Perotti (2002) and the Mountford and Uhlig (2009) identification schemes imply output elasticities of tax revenue equal to 2.3 and 3.3, respectively. Sign restrictions on impulse response functions imply output elasticities of tax revenue between 0 and 16. Different restrictions on the output elasticity of tax revenue generate a large dispersion in
the estimates of tax multipliers. For instance, I find that the impact tax multiplier is 0.2 for an output elasticity of tax revenue equal to 2.3, and 0.5 for an output elasticity of tax revenue equal to 3.3. The impact tax multiplier is negative for all output elasticities of tax revenue smaller than 1.6.

These findings lead me to the second question. I propose to measure fiscal multipliers more robustly by imposing restrictions on the output elasticities of fiscal variables in the form of probability distributions. In contrast to the existing literature, I measure these distributions both by using a variety of empirical strategies and by employing a simple dynamic stochastic general equilibrium (DSGE) model. I find that the direct measurement of prior distributions reduces the dispersion of output elasticities implied by existing identification schemes. The distribution of the output elasticity of tax revenue that I obtain ranges between 1.4 and 2. The distribution of the output elasticity of government spending ranges between -0.15 and 0.2. These restrictions are robust because they are generated by different approaches and empirical strategies and, hence, are less likely to be affected by particular assumptions or observations.

I apply this robust identification scheme to measure tax and spending multipliers associated with unexpected fiscal shocks. The estimation strategy addresses the well-known misspecification problem of SVARs in the presence of anticipated fiscal shocks (Leeper et al., 2008). I include a set of variables that react to signals about future policies, such as consumption, investment, and various measures of prices. Lagged values of these variables predict future policy actions and, consequently, help to identifying truly unexpected fiscal shocks.¹

I document three findings. First, the median impact tax multiplier is close to 0. Second, the median impact spending multiplier is 0.7 and ranges between 0.35 and 1. Third, estimates of fiscal multipliers at longer horizons are dispersed over a broad range. Despite this uncertainty, the probability

¹See Giannone and Reichlin (2006); Forni and Gambetti (2010).
that the spending multiplier is larger than the tax multiplier is above 0.8, for up to four years after policy interventions.

I also document a high probability that private consumption and real wage decline on impact following a temporary but persistent spending increase. Competing macroeconomic theories have different theoretical predictions regarding the effects of spending shocks on these variables. The standard neoclassical model (Baxter and King, 1993) predicts a decline in both variables. Standard New Keynesian models with sticky prices tend to predict a decline in consumption and an increase in real wages (Linnemann and Schabert, 2003). Finally, a recent branch of the literature (Ravn et al. (2006), Gali et al., 2007) proposes models that generate an increase in real wages and in private consumption. The evidence is in line with the standard neoclassical model.

I illustrate the framework for comparing different identification schemes with a tax policy example. Assume that only two shocks explain contemporaneous co-movements between output and tax revenue: a tax shock and a non-policy shock. The object of interest is the response of output to the tax shock. The non-policy shock controls for co-movements in the two variables due to automatic movements of tax revenue over the business cycle. In this setting, the identification of tax and non-policy shocks depends only on the restriction on one structural coefficient: the output elasticity of tax revenue.

I derive analytical relations that express tax multipliers as a function of the output elasticity of tax revenue. This parameter measures the endogenous response of tax revenue to changes in economic activity. Economists and policy-makers hold beliefs about plausible values for the output elasticity of tax revenue, formed using a variety of sources of information. For instance, national governments and international organizations estimate the output elasticity of tax revenue to construct cyclically adjusted measures of the fiscal budget. Thanks to the analytical relations, I can readily map beliefs of policy-makers and economists about plausible values of the output elasticity
Identification schemes are strategies that economists use to restrict the output elasticity of tax revenue in accordance with their prior beliefs. A simple example is the Blanchard and Perotti (2002) identification scheme, which imposes a dogmatic prior directly on the output elasticity of tax revenue. Their prior is based on the measurement of the output elasticity of tax revenue in use at the OECD. Instead, the Mountford and Uhlig (2009) identification scheme identifies tax and non-policy shocks, imposing restrictions on the sign and size of impulse response functions. These authors derive these restrictions from dynamic stochastic general equilibrium (DSGE) models. I map restrictions on impulse responses into beliefs on the output elasticity of tax revenue.

As for the robust assessment of the fiscal multipliers, following Blanchard and Perotti (2002) I use the methodology employed by the OECD (Girouard and André, 2005) as a starting point to measure output elasticities of fiscal variables. The OECD estimates output elasticities of tax revenue and spending using disaggregated data for different tax and spending categories. I integrate this methodology with additional data sources, alternative estimation techniques, and measures of elasticities based on micro-econometric studies.

I also provide measures of output elasticities of fiscal variables derived from standard DSGE models. I show that under general assumptions the output elasticities of fiscal variables are non-linear functions of deep parameters of a DSGE model. I map prior distributions on deep parameters of the DSGE model into distributions on the output elasticities of fiscal variables. I take prior distributions of deep parameters from the estimation exercise conducted in Leeper et al. (2010).

An alternative methodology for estimating the effects of fiscal policy shocks using VARs is the so-called narrative approach. Prominent examples include Romer and Romer (2010), who identify tax shocks studying narra-
tive records of tax policy decisions, and Ramey (forthcoming), who identifies
government spending shocks using changes in military spending associated
with wars. Multipliers estimated using SVAR models are different from multi-
pliers estimated using the narrative approach. My future research aims at
including narrative measures of fiscal shocks in the comparative analytical
framework developed in this paper.

The remainder of the paper is organized as follows. Section 2 describes the
identification problem faced by an econometrician who wants to identify fiscal
shocks. Section 3 derives the analytical relation between output elasticities
of tax revenue and government spending, and impulse response functions.
It also characterizes theoretical properties of the relation between output
elasticities of fiscal variables and impact multipliers. Section 4 provides a road
map of the literature, reinterpreting four different identification schemes as
restrictions on the output elasticities of fiscal variables. Section 5 describes
the robust identification scheme and reports results for impact multipliers.
Section 6 extends the analysis to multipliers at longer horizons and to output
components. Section 7 concludes the paper and suggests avenues for future
research.

3.2 The Econometric Framework

In this section I formalize the problem faced by an econometrician who wants
to identify fiscal policy shocks. Consider the reduced-form VAR model:

\[ X_t = \mu + B(L)X_{t-1} + u_t, \]

where \( X_t \) is a vector of endogenous variables, \( \mu \) is a constant, \( B(L) \) is a lag
polynomial of order \( L \), and \( u_t \) is a vector of one-step-ahead prediction errors
with mean zero and covariance matrix \( \Sigma_u = [\sigma_{ij}] \). \( \Sigma_u \) I denote the number of
variables by \( n \).

The reduced-form disturbances \( u_t \) will in general be correlated with each
3.2. **THE ECONOMETRIC FRAMEWORK**

other and consequently do not have any economic interpretation. I need to model the contemporaneous relation between reduced-form residuals $u_t$ to identify shocks $e_t$ with an economic interpretation:

$$Au_t = e_t,$$  \hspace{1cm} (3.1)

where $A$ is a matrix of structural coefficients. The structural shocks $e_t$ have mean zero and covariance matrix $\Sigma_e$. The shocks $e_t$ have an economic interpretation because they are uncorrelated with each other, i.e. $\Sigma_e$ is a diagonal matrix.

Without restrictions on the parameters in $A$, I cannot identify the structural model. The relation

$$\Sigma_u = A^{-1}\Sigma_e A^{-1'},$$

describes $n(n-1)/2$ independent equations. To solve this system I need to impose $n(n+1)/2$ restrictions on the elements of $A$.\(^2\) Without loss of generality, I normalize the diagonal elements of $A$ to unity. The additional $n(n-1)/2$ restrictions have to come from non-sample sources, as the likelihood of the model is invariant to the choice of restrictions.\(^3\)

The goal of this paper is to study how the choice of those restrictions affect the estimation of fiscal multipliers.\(^4\) From equation (3.1) I can write impact impulse responses as

$$u_t = A^{-1}e_t.$$\(^5\)

\(^2\)This is the necessary condition for exact identification stated in Rothenberg (1971). Rubio-Ramirez et al. (2010) derive necessary and sufficient conditions for global identification of exactly identified models which, in addition to the Rothenberg (1971) counting condition, require that restrictions follow a certain equation by equation pattern. The SVAR studied in this paper satisfy these conditions for global identification.

\(^3\)A more detailed description of the econometric framework is provided in appendix A.1. For a detailed discussion of Structural VAR models see Lütkepohl (2005).

\(^4\)Fiscal multipliers are rescaled responses of output to fiscal shocks.
Columns of matrix \( A^{-1} \) are known as impulse vectors (Uhlig (2005)). The effect on variable \( i \) of shock \( j \) is the \( i,j \) element of matrix \( A^{-1} \). In the next section I derive an analytical relation between impact responses and identification restrictions. Analytical expressions for impact multipliers are simple to analyze, as they only depend on the coefficients of the reduced-form covariance matrix \( \Sigma_u \) and on the restricted coefficients in matrix \( A \). I characterize theoretical properties of impact responses that hold for any covariance matrix \( \Sigma_u \). However, I do not analytically characterize theoretical properties of impulse responses at longer horizons, as they also depend on the coefficients of the reduced-form lag polynomial \( B(L) \). In section 3.6 I show how analytical results facilitate a numerical analysis of impulse responses at longer horizons.

For simplicity and clarity, I start in section 3.3 by discussing separately the analytical identification of tax shocks and spending shocks. To illustrate the analytical results, as well as to conduct the empirical analysis of fiscal multipliers, I first estimate a bivariate tax model and a bivariate spending model consisting of the policy variable and output. Then I estimate 11-equation models. I add a block of 9 forward-looking variables to the basic bivariate models. I estimate all models using Bayesian techniques. Models include a constant and six lags. The data are quarterly and range from 1947:1 to 2010:1. Appendix A.3 provides a more detailed description of data and methodology.¹

### 3.3 The Analytics of Identification

To understand how the choice of restrictions affects inference, I reduce the dimensionality of the problem to its essence. In fiscal applications, I can characterize the identification problem using bivariate models. I assume that the

¹Caldara and Kamps (2008) explore how the choice of reduced-form model affects the estimates of fiscal multipliers.
model consists of a non-policy variable that is ordered first in the VAR system and of a policy instrument that is ordered second. The non-policy variable is the logarithm of output \((Y_t)\) in real, per-capita terms. The policy instrument, denoted by \(P_t\), is either tax revenue\(^6\) \((T_t)\), or government consumption and investment \((G_t)\), both in real, per-capita terms.

The relation between reduced-form disturbances \(u_t\) is:

\[
\begin{align*}
    u_{Y,t} &= a_{Y,P}u_{P,t} + e_{Y,t} & (3.2) \\
    u_{P,t} &= a_{P,Y}u_{Y,t} + e_{P,t}, & (3.3)
\end{align*}
\]

where \(a_{Y,P}\) and \(a_{P,Y}\) are elements of the matrix \(A\):

\[
A = \begin{bmatrix}
    1 & -a_{Y,P} \\
    -a_{P,Y} & 1
\end{bmatrix}.
\]

Equation (3.2) states that unexpected movements in output are due to unexpected movements in policy \((a_{Y,P}u_{P,t})\) or to sources of business cycle fluctuations unrelated to the policy under investigation \((e_{Y,t})\). Equation (3.3) states that unexpected changes in policy are either endogenous to the business cycle \((a_{P,Y}u_{Y,t})\), or exogenous to the business cycle \((e_{P,t})\) and uncorrelated with non-policy sources of fluctuations. Endogeneity of policy can arise either because policy-makers react to contemporaneous developments in economic activity, or because of the automatic feedback from economic activity to tax revenue and government spending. In this paper, I follow Blanchard and Perotti (2002), B&P henceforth, and assume that the first channel is eliminated by the use of quarterly data. This is plausible due to information lags, legislative lags, and implementation lags faced by fiscal policy-makers.\(^7\)

\(^6\)As in B&P and Mountford and Uhlig (2009), I treat government transfers to persons as negative taxes.

\(^7\)Lags in the legislation and implementation of fiscal policy actions might lead to the anticipation by economic agents of future policy actions. A growing number of papers studies the implications of this phenomenon, known a fiscal foresight, for the estimation of the effects of unanticipated shocks (e.g., Leeper et al. 2008). In the simple bivariate
Consequently the coefficient $a_{21}$ captures the automatic response of fiscal variables to changes in economic activity, measured as the output elasticity of tax revenue ($\eta_{T,Y}$) and of government spending ($\eta_{G,Y}$).

In order to identify the SVAR model I impose a restriction on $a_{P,Y}$. To highlight the restricted coefficient, I denote throughout the paper $a_{P,Y}$ as $\eta_{P,Y}$. In the public finance literature, a large body of research measures the output elasticity of fiscal variables. The output elasticity of tax revenue $\eta_{T,Y}$ is the most familiar measure of sensitivity of taxes to income changes. This elasticity serves as an indicator of the tax system’s overall progressivity. A proportional income tax has an elasticity of 1.0. Progressive tax systems, where tax-income ratios increase with income, have an elasticity greater than 1.0. As far as output elasticity of spending $\eta_{G,Y}$ is concerned, most studies assume its value to be zero, based on the observation that government consumption and investment have weak cyclical components. As I show in Section 3.5, recent empirical studies depart from this assumption and attempt to estimate $\eta_{G,Y}$.

To produce inference, econometricians need to impose a numerical restriction on $\eta_{P,Y}$. Numerical restrictions as priors of the econometrician regarding a plausible value, or a set of plausible values, for the elasticities. As I describe in Section 3.4, in the SVAR literature econometricians have formed and implemented priors on elasticities using a variety of methods.

The system described by (3.2) and (3.3) can also be written in terms of impulse vectors as:

$$
\begin{bmatrix}
    u_{Y,t} \\
    u_{P,t}
\end{bmatrix} = \frac{1}{1 - a_{Y,P} \eta_{P,Y}} \begin{bmatrix}
    1 & a_{Y,P} \\
    \eta_{P,Y} & 1
\end{bmatrix}_A^{-1} \begin{bmatrix}
    e_{Y,t} \\
    e_{P,t}
\end{bmatrix}.
$$

model, I abstract from fiscal foresight, which I discuss in Section 3.1.

---

8As explained in the previous section, I need to impose $n(n - 1)/2$ restrictions to have an exactly-identified SVAR. In a bivariate model this amounts to imposing only one restriction.
3.3. THE ANALYTICS OF IDENTIFICATION

The object of interest is the response of output to a policy shock:

$$A_{Y,P}^{-1} = \frac{a_{Y,P}}{1 - a_{Y,P} \eta_{P,Y}}. \quad (3.4)$$

In particular, I want to study how $A_{Y,P}^{-1}$ depends on the beliefs of econometricians about $\eta_{P,Y}$. The main difficulty is that the coefficient $a_{Y,P}$ depends on both $\eta_{P,Y}$ and on the reduced-form coefficients $\Sigma_u$. In the bivariate model there exists a simple closed-form solution. Denote the elements of the variance-covariance matrix $\Sigma_u$ as

$$\Sigma_u = \begin{bmatrix} \sigma_{YY} & \sigma_{YP} \\ \sigma_{YP} & \sigma_{PP} \end{bmatrix}.$$

The solution for $a_{12}$ is:

$$a_{Y,P}(\eta_{P,Y}; \Sigma_u) = \frac{\sigma_{YP} - \eta_{P,Y} \sigma_{YY}}{\sigma_{PP} - \eta_{P,Y} \sigma_{YP}}.$$

I substitute this expression for $a_{Y,P}$ into (3.4) so as to re-write the output response to a policy shock as:

$$A_{Y,P}^{-1}(\eta_{P,Y}, \Sigma_u) = \frac{\sigma_{YP} - \eta_{P,Y} \sigma_{YY}}{\eta_{P,Y} \sigma_{YY} + \sigma_{PP} - 2\eta_{P,Y} \sigma_{YP}}. \quad (3.5)$$

For a given choice of the reduced-form model ($\Sigma_u$), I analyze the output response to a policy shock as a function of the identification restriction on the output elasticity of the policy variable. The following proposition states key properties of the output response to the policy shock.

**Proposition 1.** The output response to a policy shock (3.5) has the following properties:

---

9. I also derive the analytical solution for $\Sigma_e$, reported in the Appendix.

10. The assumption that $\Sigma_u$ is positive definite ensures that the denominator of (3.5) is strictly larger than zero. This guarantees that impulse response functions are defined for all output elasticities $\eta_{P,Y}$. 

1. It has a global minimum and a global maximum such that:

\[ A_{Y,P}^{-1}(\eta_{P,Y}^{\text{min}}, \Sigma_u) < 0 \]
\[ A_{Y,P}^{-1}(\eta_{P,Y}^{\text{max}}, \Sigma_u) > 0 \]

where \( \eta_{P,Y}^{\text{min}} = \arg \min_{\eta_{P,Y}} A_{Y,P}^{-1}(\eta_{P,Y}, \cdot) \), \( \eta_{P,Y}^{\text{max}} = \arg \max_{\eta_{P,Y}} A_{Y,P}^{-1}(\eta_{P,Y}, \cdot) \), and \( \eta_{P,Y}^{\text{max}} < \eta_{P,Y}^{\text{min}} \).

2. It equals zero if and only if \( \eta_{P,Y} = \sigma_{YP}/\sigma_{YY} \equiv \bar{\eta}_{P,Y} \).

3. It is strictly decreasing for \( \eta_{P,Y} \in [\eta_{P,Y}^{\text{max}}, \eta_{P,Y}^{\text{min}}] \), and strictly increasing for \( \eta_{P,Y} < \eta_{P,Y}^{\text{max}} \) or \( \eta_{P,Y} > \eta_{P,Y}^{\text{min}} \).

**PROOF:** See Appendix.

The first part of Proposition 1 states that the set of admissible output responses to a policy shock is bounded. These bounds have opposite signs. If the econometrician does not have any information to limit the set of plausible values for the elasticity \( \eta_{P,Y} \), the sign of the output response cannot be determined. The first and second part of the proposition imply that the output response is positive for all \( \eta_{P,Y} < \eta_{P,Y}^{\text{max}} \), while it is negative for all \( \eta_{P,Y} > \eta_{P,Y}^{\text{min}} \). The third part characterizes how changes in the elasticity affect the impact response.

I use Proposition 1 to study the identification problem in the bivariate tax and spending models. I first rescale the impulse response as a multiplier, reporting the dollar change in output in response to a fiscal shock of size one dollar. To facilitate the comparison between tax multipliers and spending multipliers, I compare shocks that are intended to stimulate output. For that purpose, I analyze the effects of structural tax cuts but structural spending increases.

The impact tax (cut) multiplier is
\[ \Pi_{0}^{T,Y} (\eta_{T,Y}; \Sigma_u) = -A_{Y,P}^{-1} (\eta_{T,Y}; \Sigma_u) \frac{1}{T/Y}, \]

where \( T/Y \) is the sample mean of the tax-to-output ratio.\(^{11}\) The top panel of Figure 1 plots \( \Pi_{0}^{T,Y} \) as a function of the output elasticity of tax revenue. I evaluate the variance-covariance matrix \( \Sigma_u \) at the OLS estimates from the tax model.

The \textit{impact spending multiplier} is

\[ \Pi_{0}^{G,Y} (\eta_{G,Y}; \Sigma_u) = A_{Y,P}^{-1} (\eta_{G,Y}; \Sigma_u) \frac{1}{G/Y}, \]

where \( G/Y \) is the sample mean of the spending-to-output ratio. The bottom panel of Figure 1 plots \( \Pi_{0}^{G,Y} \) as a function of the output elasticity of spending. I evaluate the variance-covariance matrix \( \Sigma_u \) at the OLS estimates from the spending model.

Table 1 the bounds for the tax and spending multipliers and the key values for the elasticities. If I do not have any information to limit the set of admissible elasticities, I still know that the tax multiplier ranges between \( \pm 0.69 \) dollars, while the spending multiplier ranges between \( \pm 1.54 \) dollars. However, typically I may have some information to narrow down the set of plausible elasticities, although I am uncertain about the exact values. I can then use Proposition 1 to learn whether, given some uncertainty on elasticities, I can identify the sign of multipliers. Elasticities of output with respect to taxes smaller than 1.50 are associated with negative tax multipliers, while elasticities larger than 1.50 are associated with positive multipliers.\(^{12}\) Similarly, elasticities of spending smaller than 0.29 are associated with positive

\(^{11}\)This scaling factor converts percent changes into dollar changes, the latter being the unit in which multipliers are usually reported. I evaluate fiscal multipliers at the sample mean tax-to-output ratio, as in B&L and Mountford and Uhlig (2009).

\(^{12}\)Remember that I study a negative tax shock, i.e I multiply \( A_{Y,P}^{-1} \) by \((-1)\). I mirror \( A_{Y,P}^{-1} \) over the \( x-\)axes.
spending multipliers, while elasticities larger than 0.29 are associated with negative multipliers. Finally, I know that for output elasticities of tax revenue that range between $\eta_{T,Y}^{\text{min}} = -3.07$ and $\eta_{T,Y}^{\text{max}} = 6.07$, the tax multiplier is increasing in the elasticity. If I have non-sample information that the elasticity lies between 1, describing a proportional tax system, and $\eta_{T,Y}^{\text{max}}$, describing an extremely progressive tax system, I know that the tax multiplier will be at least $-0.15$ dollars.

### 3.3.1 Multivariate Models

I now characterize analytically impact impulse responses in VAR models with more than 2 variables. I derive the analytical framework in Appendix A.1. This generalization is useful for three reasons. First, I need additional variables to replicate some identification schemes applied in the literature and analyzed in Section 3.4. Second, I want to study the effects of policy shocks on other variables, as illustrated in Section 3.6 for private consumption, investment, and real wages. Third, bivariate VAR models may omit variables that can predict output and the policy variable of interest. Leeper et al. (2008) point out that, because of legislative and implementation lags, agents could receive signals about future fiscal policy changes, a phenomenon known as fiscal foresight. In the presence of fiscal foresight, agents react to changes in policy before its actual implementation. If the information set of the econometrician is not aligned with the information set of the agents, SVARs can produce distorted inference about the effects of policies. One solution to this problem is to include a large set of forward-looking variables in the VAR. If agents truly react to signals, lagged values of consumption, investment, and prices should predict future policy actions. This ensures that the identified shocks $e_t$ are truly unanticipated.13

13Leeper et al. (2008, 2009b) show that simple macroeconomic models where agents receive signals about future fiscal policy do not have a VAR representation. These models are non-invertible. Giannone and Reichlin (2006) and Forni and Gambetti (2010) suggest that forward-looking variables should mitigate the non-invertibility problem. If the econometri-
In VAR models with \( n \) equations, the identification of policy shocks \( e_{P,t} \) depends on \( n - 1 \) structural coefficients. In a multivariate model, equation (3.3) becomes:

\[
e_{P,t} = u_{P,t} - \eta_{P,Y} u_{Y,t} - a_{P,3} u_{3,t} - \ldots - a_{P,n} u_{n,t}.
\] (3.6)

Consequently, the impulse vector associated with \( e_{P,t} \) would depend on all structural coefficients appearing in equation (3.6). To ensure that the response of all endogenous variables to \( e_{P,t} \) depends only on \( \eta_{P,Y} \), I assume that additional variables in the VAR do not affect contemporaneously the policy instrument:

\[
a_{P,i} = 0, \text{ for } i = 3, \ldots n.
\]

Furthermore, I assume that additional variables in the VAR do not affect contemporaneously output. This assumption ensures that also the impact response of variables to the non-policy shock \( e_{Y,t} \) is only a function of the elasticity \( \eta_{P,Y} \).\(^{14}\) I summarize the assumptions on matrix \( A \) in the following.

**Assumption 1.** If \( n \geq 3 \), matrix \( A \) is block recursive:

\[
A = \begin{bmatrix}
1 & -a_{Y,P} & 0 \\
-\eta_{P,Y} & 1 & 0 \\
\vdots & \vdots & \vdots \\
Block 2
\end{bmatrix},
\]

where block 2 is an \((n - 2 \times n)\) submatrix of structural coefficients for the additional variables in the VAR, and 0 is an \(1 \times n - 2\) vector of zeros.

Under Assumption 1 I isolate the crucial dimension of the identification

cian observes a large number of forward-looking variables, the model should become close to invertible, and the bias in inference should be small. For a detailed discussion of non-invertibility see Sims (1988) and Fernandez-Villaverde et al. (2007). For the identification of anticipated spending shocks in SVAR models, see Mertens and Ravn (2010).

\(^{14}\)The non-policy shock \( e_{Y,t} \) plays a crucial role in the analysis of the sign restriction approach in section 3.4.
problem, the causal relation between the policy instrument and output. By relaxing Assumption 1 I could study how the contemporaneous interaction between the policy variable and other variables in the system affect the identification of the policy shock and the size of fiscal multipliers. As shown in Appendix A.1, I can derive analytical expressions for impulse vectors that depend on more than one structural parameter $a_2$. However for the identification of fiscal shocks Assumption 1 does not seem very restrictive. B&P base their analysis of fiscal multipliers on Assumption 1. Furthermore, they show that the modeling of the contemporaneous relation between tax revenue and spending has effect on the estimates of fiscal multipliers, as the correlation between spending and tax residuals is close to zero. Similarly, Perotti (2005) finds that the modeling of the contemporaneous relation between fiscal variables, inflation, and interest rate has also little effect on the estimates of fiscal multipliers. Based on this evidence, I do not explore these interactions in this paper. Moreover, the set of multipliers I characterize varying $\eta_{P,Y}$ under Assumption 1 is a subset of all admissible multipliers I could characterize relaxing this assumption.

Finally, notice that Assumption 1 implies that shocks $e_{i,t}$, for $i = 3, \ldots, n_x$ do not affect contemporaneously $Y_t$ and $P_t$:

$$A_0^{-1} = \begin{bmatrix} A_{Y,Y}^{-1} & A_{Y,P}^{-1} & 0 \\ A_{P,Y}^{-1} & A_{P,P}^{-1} & 0 \\ \hline \hline Block 2 \end{bmatrix}.$$  

The non-policy shock $e_{Y,t}$ and the policy shock $e_{P,t}$ explain all contemporaneous variability in $Y_t$ and $P_t$. The identification of the remaining shocks does not affect the inference on the fiscal multipliers. This means that the analytical expressions for the response of output and tax revenue to shocks $e_{Y,t}$ and $e_{P,t}$ are identical to the expressions derived in the bivariate model. As shown in Appendix A.1, Assumption 1 facilitates the analytical the char-
3.3. THE ANALYTICS OF IDENTIFICATION

Characterization of the response of any variable in the VAR to shocks $e_{Y,t}$ and $e_{P,t}$.\footnote{I analyze responses of other variables to these shocks in sections 3.4 and 3.6.} Of course, this is a simplifying assumption, and I base it on the idea that one shock ($e_{Y,t}$) is enough to control for co-movements in $Y_t$ and $P_t$ unrelated to the policy of interest. Mountford and Uhlig (2009) identify, in addition to a non-policy shock, a monetary policy shock, and they find that the identification of this shock has a small impact on the fiscal multipliers. I interpret this evidence as supportive of Assumption 1.

Figure 2 compares the impact tax and spending multipliers, when these are estimated using bivariate (black dashed line) and 11-equation (blue solid line) tax and spending models.\footnote{Refer to Appendix A.3 for additional details.} The first variable in the 11-equation models is output. The second variable is either tax revenue or spending. The third variable in the system is tax revenue in the spending model, or spending in the tax model. Tax revenue is simply a control variable in the spending model and vice versa. The remaining variables are a set of real variables (private consumption, residential and non-residential investment), and a set of prices (CPI, PPI, a stock market index, short-term interest rate, real wage).

Impact tax multipliers for the bivariate and the 11-equation models are similar. This is because the additional variables in the VAR do not change substantially the covariance structure between output and fiscal residuals. Yang (2007) argues that the use of forward-looking variables should also detect the incidence of fiscal foresight. The similarity between the estimates in the 2-equation and 11-equation models suggests that fiscal foresight does not seem to bias impact multipliers.

Summing up, I have showed that the sign and size of spending and tax multipliers depend on the choice of the output elasticity of tax revenue and government spending. I have also characterized analytically the identification problem. In the next section, I show how a number of different identification schemes used in the existing literature can be mapped into priors of econo-
metricians regarding the output elasticities of fiscal variables.

3.4 The Literature Road Map

In this section, I use analytical results to reinterpret identification schemes as priors on output elasticities of fiscal variables. I show that these priors are different enough to produce widely divergent fiscal multipliers.

Some theoretical results depend on the sign of the correlation coefficient between the reduced-form residuals of output and the policy variable. Throughout the paper, I assume that output and policy residuals are positively correlated.

Assumption 2. For $P = T, G$:

$$\text{Corr}(u_{Y,t}, u_{P,t}) = \rho_{YP} > 0.$$ 

This assumption is satisfied not only in VAR models estimated in this paper, but in most empirical fiscal VAR models.$^{17}$ I examine four identifications schemes used in the literature: the recursive approach, the traditional SVAR analysis implemented by B&P, the “pure” sign restriction approach, and the penalty function approach to sign restrictions. I summarize numerical results in Tables 2, 3, and 4, and in Figure 3. The analysis focuses on impact fiscal multipliers, but the comparison could be easily extended to fiscal multipliers at longer horizons.

3.4.1 The Recursive Approach

I analyze first the recursive approach, proposed by Sims (1980). In the standard implementation, the recursive approach restricts the matrix of impulse vectors $A^{-1}$ to a lower triangular matrix, which implies a causal ordering of

$^{17}$For example, the reduced-form models estimated by B&P and Mountford and Uhlig (2009) satisfy Assumption 2.
model variables. If $A^{-1}$ is lower triangular, the first variable in the VAR only reacts to shock $e_{1,t}$, the second variable only reacts to shocks $e_{1,t}$ and $e_{2,t}$, and so on.

Assumption 1 implies that output and the policy variable do not react contemporaneously to shocks $e_{i,t}$, for $i = 3, ..., n$. This assumption implies that output and the policy variable of interest are ordered before variables 3 to $n$. Hence I only need to explore the two possible orderings between output and the policy variable.

I first order output before the policy variable. This ordering means that output does not react to a policy shock $e_{P,t}$:

$$
\begin{bmatrix}
  u_{Y,t} \\
  u_{P,t}
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 \\
  A^{-1}_{Y,P} & 1
\end{bmatrix}
\begin{bmatrix}
  e_{Y,t} \\
  e_{P,t}
\end{bmatrix}.
$$

As I know from Statement 2 of Proposition 1, the response of output to a policy shock is zero if and only if $\eta_{Y,P} = \eta_{Y,P} = \sigma_{YP}/\sigma_{YY}$. I can then translate the prior of the econometrician that the response of output to tax shocks is zero, into a prior that the output elasticity of tax revenue is $\eta_{T,Y} = 1.62$. Similarly, the prior on the output elasticity of government spending consistent with a zero response of output to a spending shock is $\eta_{G,Y} = 0.40$.

Alternatively, I order output after the policy variable. This ordering implies that the policy instrument does not react to a non-policy shock $e_{Y,t}$. Instead of changing the ordering of variables as in the standard analysis, I just impose this restriction in the matrix $A^{-1}$:

$$
\begin{bmatrix}
  u_{Y,t} \\
  u_{P,t}
\end{bmatrix} =
\begin{bmatrix}
  1 & A^{-1}_{Y,P} \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  e_{Y,t} \\
  e_{P,t}
\end{bmatrix}.
$$

I can solve for the values of $\eta_{P,Y}$ that satisfy $A_{P,Y}^{-1}(\eta_{P,Y}, \Sigma_u) = 0$. The only admissible solution to this equation is $\eta_{P,Y} = 0$. Plugging this solution into
the expression for the output response to the policy shock (3.5) I obtain:

\[ A_{P,Y}^{-1}(\eta_{P,Y}, \Sigma_u) = \frac{\sigma_{Y,P}}{\sigma_{PP}}. \]

How large are the impact multipliers? The impact tax multiplier is equal to \(-\sigma_{YT}/\sigma_{TT}\) (recall that I study tax cuts, i.e. shocks of size \(-1\)) which equals to \(-0.41\) dollars. The tax multiplier becomes negative because an output elasticity of tax revenue of zero means that the non-policy shock does not generate any co-movement between output and tax revenue. The positive co-movement observed in the data must thus be generated by the policy shock alone. If taxes are cut, output decreases.

The impact spending multiplier is 0.70 cents. Since output and spending are also positively correlated, the spending shock must generate a positive co-movement between output and spending equal to what I observe in the data.

Are spending multipliers larger than tax multipliers according to the recursive approach? To answer this question, I need to select a recursive ordering among the two orderings analyzed above. I identify spending shocks following the second recursive ordering. The assumption that government spending does not react to movements in output has been largely used in the literature, most prominently by Blanchard and Perotti (2002) and Fatas and Mihov (2001). The spending multiplier, which I plot in the bottom panel of Figure 3 is therefore 0.70. I identify tax shocks imposing the second ordering. The choice of recursive ordering to identify tax shocks is more problematic, since both orderings seem to be too restrictive. On the one hand, I do not want to impose that the impact tax multiplier is equal to zero. On the other hand, imposing that tax revenue does not react automatically to the business cycle seems unreasonable. I select the first ordering because I know that a zero impact tax multiplier is associated with an elasticity of 1.62, a value of
the elasticity that seems empirically plausible.\textsuperscript{18} I provide a measure of uncertainty around the point estimates in Table 2 for the tax multiplier and in Table 3 for the spending multiplier. As shown in Table 4, the probability that the impact spending multiplier to be larger than the impact tax multiplier is 1.

\section*{3.4.2 Traditional SVAR}

The identification approach adopted by B&P relies on institutional information about the tax and transfer systems and about the timing of tax collections in order to identify the automatic response of tax revenue and spending to economic activity. B&P form their prior about plausible elasticities based on off-model information. I provide a detailed analysis of the B&P methodology in Section 3.5. I apply the B&P methodology to obtain point estimates for the elasticities for the sample 1947 – 2010. The output elasticity of tax revenue is 2.26, while the output elasticity of government spending is zero. Thus, B&P believe that the output elasticity of tax revenue is larger than do econometricians using the recursive approach. This difference leads to a larger tax multiplier (0.20) than the recursive approach (0). As for the output elasticity of spending, econometricians using the B&P and the recursive approach have a prior that the this elasticity is zero. As I saw in the previous section, the resulting spending multiplier is 0.70. I plot these multipliers in Figure 3 and provide measures of uncertainty around the point estimates in Tables 2 and 3. Traditional SVAR analysis, as the recursive approach, estimates an impact spending multiplier below 1. The probability of the spending multiplier to be larger than the tax multiplier is 0.99, as reported in Table 4. In other words, differences in beliefs about the elasticities between B&P and users of the recursive approach would not change the answer to the question of which multiplier is the largest.

\textsuperscript{18}See Section 3.5 for details.
3.4.3 “Pure” Sign-Restriction Approach

An alternative approach to identification is to impose sign restrictions on impulse responses. I base the discussion of this approach on the work by Mountford and Uhlig (2009).\textsuperscript{19} For the sake of simplicity, I only impose sign restrictions on impact responses. The framework can be extended to impose restrictions at longer horizons. Importantly, the tax and spending model now need to be analyzed separately.

3.4.3.1 Tax Model

The sign restriction approach is a partial identification method. It does not require to identify all shocks in the SVAR, but only the shocks of interest.\textsuperscript{20} As already explained in Section 3.3, I only identify two shocks: The non-policy shock $e_{Y,t}$, which captures cyclical movements in output and the policy variable, and the policy shock $e_{P,t}$, which is the shock of interest. Compared to Mountford and Uhlig (2009), the additional assumption I impose is that the identification of these two shocks only depends on the output elasticity of the policy variable $\eta_{P,Y}$. The sign restriction approach imposes qualitative restrictions directly on impulse responses, i.e. on the elements of matrix, which are motivated by economic theory. Let us consider a simple example. Assume that a non-policy shock increases output and tax revenue on impact. A large class of micro-founded macroeconomic models (e.g., Forni et al. 2009) agrees that this sign pattern is consistent with supply shocks (e.g., technology shocks) and demand shocks (e.g., preference shocks). Furthermore, assume that a tax shock is a shock that increases tax revenue on impact. This restriction follows Mountford and Uhlig (2009), who argue that theoretical models disagree regarding the response of other macroeconomic

\textsuperscript{19}M&U impose sign restrictions on impulse responses in combination with a criterion function, discussed in the next sub-section. The exercise in this section unveils what the inference on fiscal multipliers in M&U without penalty function would have been.

\textsuperscript{20}For details see Uhlig (2005).
variables to tax shocks. As in Mountford and Uhlig (2009), I will maintain this assumption throughout the paper. I restrict the elements of matrix $A^{-1}$ as follows:

$$A^{-1} = \begin{bmatrix} + & ? & \cdots \\ + & - & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$ 

Using the analytical expression for $A^{-1}$, I characterize the set of all output elasticities of tax revenue $\eta_{T,Y}$ that satisfy this sign pattern. Properties of this set depend on the sign of the correlation coefficient between the residuals of output and the policy instrument. I base the discussion on Assumption 2, i.e., that the correlation coefficient between output and tax revenue residuals is positive.

Notice that the restriction that tax revenue has to decrease following a tax shock is simply a sign normalization, because I want to identify tax cuts. This normalization does not rule out any value of $\eta_{T,Y}$. In fact, an elasticity $\eta_{T,Y}$ such that the response of tax revenue to a tax shock is positive is consistent with a tax increase. I obtain a shock $e_{T,t}$ that satisfies the sign restriction simply multiplying the candidate shock by $-1$, i.e., I transform a tax increase into a tax cut. Consequently, only the sign restrictions imposed on the response of variables to a non-policy shock restrict the set of admissible elasticities.

**Proposition 2.** Assume that a non-policy shock $e_{T,Y}$ is a shock that increases output and tax revenue on impact. Let $H_{T,Y}^{SR}$ be the set of output elasticities of tax revenue $H_{T,Y}^{SR}$ that satisfy the sign restrictions. Under Assumption 2:

$$H_{T,Y}^{SR} \equiv \{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} > 0 \}.$$

**PROOF:** See Appendix.

Imposing sign restrictions on the impact response of output and tax rev-
enue restricts the set of admissible elasticities to positive values. Imposing additional sign restrictions on the impact response of other variables might restrict the set of admissible elasticities.

**Proposition 3.** Assume that non-policy shock $e_{Y,t}$ is a shock that increases output, tax revenue, and variable $i$, for $i = 3, \ldots, n_x$, on impact. Let $H_{T,Y}^{SR}$ be the set of output elasticities of tax revenue $\eta_{T,Y}$ that satisfy the sign restrictions. Under Assumptions 1, and 2 there are two cases:

**CASE 1.**

If $\rho_{Ti} > 0$, $\rho_{Ti} - \rho_{YT} \rho_{Yi} > 0$, and $\rho_{YT} \rho_{Ti} - \rho_{Yi} < 0$, then:

$$H_{T,Y}^{SR} \equiv \{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} > 0 \},$$

for $i = 3, \ldots, n$.

**CASE 2.**

If $\rho_{Ti} < 0$, $\rho_{Ti} - \rho_{YT} \rho_{Yi} < 0$, and $\rho_{YT} \rho_{Ti} - \rho_{Yi} < 0$ then:

$$H_{T,Y}^{SR} \equiv \left\{ \eta_{T,Y} \in \mathbb{R} : 0 < \eta_{T,Y} < \frac{\sigma_T (\rho_{YT} \rho_{Ti} - \rho_{Yi})}{\sigma_Y (\rho_{Ti} - \rho_{YT} \rho_{Yi})} \right\},$$

for $i = 3, \ldots, n$.

**PROOF:** See Appendix.

Proposition 3 shows that imposing restrictions on additional variables may or may not narrow down the set of admissible tax elasticities $H_{T,Y}^{SR}$ depending on the correlation pattern between the restricted variables.\textsuperscript{21} All variables in the VAR but one\textsuperscript{22} follow the correlation patterns assumed in Proposition 3.

\textsuperscript{21} I designed Proposition 3 to analyze the variables included in the 11-equation VAR model estimated in this paper. The proof of Proposition 3 provided in Appendix A.1 can be extended to analyze different correlation patterns.

\textsuperscript{22} The Consumer Price Index violates the condition $\rho_{YT} \rho_{Ti} - \rho_{Yi} < 0$, which is 0.0035.
Six variables in the 11-equation VAR satisfy the correlation pattern required by Case 1: government spending, private consumption, residential and non-residential investment, the 3-month interest rate on government bonds, and real wages. Positive restrictions on most of these variables would be consistent with the effects of a technology shock in a large class of DSGE models. At the same time, imposing these restrictions would not narrow down the set of admissible output elasticities of tax revenue. For instance, Mountford and Uhlig (2009) identify the non-policy shock imposing restrictions on output, tax revenue, private consumption, and non-residential investment. Proposition 3 shows that the set of elasticities satisfying these restrictions ranges between 0 and infinity. Consequently, the set of admissible tax multipliers would range between $-0.41$ and 0.69 dollars.

Two variables satisfy the correlation pattern required by Case 2: the stock price index and the Producer Price Index (PPI). Output elasticities of tax revenue should be between 0 and 11 in order to generate a positive response of output, tax revenue, and the stock market index to a non-policy shock. Elasticities between 0 and 11 generate a positive response of output, tax revenue, and the stock prices. This set of restrictions is in line with the sign pattern generated by non-policy shocks in DSGE models (e.g., Smets and Wouters 2007). At the same time the set of elasticities that generate this sign pattern is large, and it modestly helps to narrow down the set of admissible impact tax multipliers, that ranges between $-0.41$ and 0.69. Output elasticities of tax revenue should be between 0 and 1.55 to generate a positive response of output, tax revenue, and PPI. This sign pattern would provide sharp inference of impact tax multipliers. The problem is that, as in Mountford and Uhlig (2009), I cannot impose this restriction. The non-policy shock captures all movements in endogenous variables due to demand and supply shocks orthogonal to fiscal policy. In DSGE models, PPI declines in response to supply shocks, while it increases after demand shocks. A similar reasoning applies to CPI, the only variable in the VAR that does not satisfy
assumptions in Proposition 3.

3.4.3.2 Spending Model

Similar to the identification of the tax shock, Mountford and Uhlig (2009) identify the spending shock as a shock that increases spending. This restriction, if imposed only on impact, simply normalizes the sign of the shock. As in the tax model, I need to concentrate on the identification of the non-policy shock. The tax and spending models differ under one crucial dimension: Mountford and Uhlig (2009) do not impose any restriction on the response of government spending to the non-policy shock. In fact, as already argued in the previous section, they identify a non-policy shock as a shock that increases output, tax revenue, consumption, and non-residential investment on impact.

Proposition 4. Assume that non-policy shock is a shock that increases output and variable $i$, for $i = 2, ..., n$, on impact. Let $H_{G,Y}^{SR}$ be the set of output elasticities of spending $\eta_{G,Y}$ that satisfy the sign restrictions. Under Assumptions 1 and 2 there are two cases:

**CASE 1.**

If $\rho_{Gi} < 0$, and $\rho_{Gi} - \rho_{YG}\rho_{Yi} < 0$, then:

$$H_{G,Y}^{SR} \equiv \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} < \frac{\sigma_{G}(\rho_{YG}\rho_{Gi} - \rho_{Yi})}{\sigma_{Y}(\rho_{Gi} - \rho_{YG}\rho_{Yi})} \right\},$$

for $i = 3, ..., n$.

**CASE 2.**

If $\rho_{Gi} > 0$, $\rho_{Gi} - \rho_{YG}\rho_{Yi} > 0$, then:

$$H_{G,Y}^{SR} \equiv \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} > \frac{\sigma_{G}(\rho_{YG}\rho_{Gi} - \rho_{Yi})}{\sigma_{Y}(\rho_{Gi} - \rho_{YG}\rho_{Yi})} \right\},$$
for $i = 3, \ldots, n$.

**PROOF:** See Appendix.

Five variables satisfy the correlation pattern required by Case 1: private consumption, residential and non-residential investment, the 3-month interest rate on government bonds, and real wages. Let us analyze the two variables restricted by Mountford and Uhlig (2009): private consumption and non-residential investment. The response of output and consumption to a non-policy shock is positive if the output elasticity of government spending is smaller than 6.28. The response of output and non-residential investment to a non-policy shock is positive if the output elasticity of government spending is smaller than 5.76. If I restrict output, private consumption and non-residential investment to be positive, the output elasticity of spending must be smaller than 5.76. This is a very loose restriction on $\eta_{G,Y}$, as empirically plausible values for this elasticity are in a neighborhood of zero.

Three variables satisfy the correlation pattern required by Case 2: tax revenue, PPI, and CPI. Let us analyze the variable restricted by Mountford and Uhlig (2009), tax revenue. The response of output and tax revenue to a non-policy shock is positive if the output elasticity of government spending is larger than $-59$. As already mentioned in the analysis of tax shocks, I cannot restrict the response of PPI and CPI to non-policy shocks, as the response of these variables to demand and supply shocks in DSGE models have opposite sign.

The two cases combined allow us to determine the set of output elasticities of spending that satisfy the sign restrictions imposed by Mountford and Uhlig (2009) on output, tax revenue, private consumption, and non-residential investment. This set includes all elasticities between $-59$ and 5.76. These restrictions do not narrow the set of admissible spending multipliers, that ranges between the lower and upper bounds characterized in Proposition 1 ($\pm 1.52$).

Summing up, imposing sign restrictions on impact responses corresponds
to very loose priors of the econometricians regarding the plausible size of output elasticities of fiscal variables. These priors does not allow to obtain sharp inference on fiscal multipliers, as shown in Tables 2 and 3. The probability that the spending multiplier is larger than the tax multiplier is 0.44.

3.4.4 "Penalty-Function" approach to Sign-Restriction

Mountford and Uhlig (2009) acknowledge that imposing sign restrictions on impulse responses does not produce sharp inference on fiscal multipliers.\(^{23}\)

For this reason they decided to impose an additional identification restriction. Their starting point is the observation that in a large class of DSGE models tax and spending shocks contribute little to business cycle fluctuations. Non-policy shocks should explain most of the variability in the data. Mountford and Uhlig (2009) translate this observation into an identification restriction setting up a criterion function that ascribes as much movement as possible to the non-policy shock. The selected shock has also to satisfy the sign restrictions discussed in the previous section.\(^{24}\)

I think about the penalty function as selecting one elasticity from the set of elasticities \(H_{SR}^{P,Y}\) that satisfy sign restrictions. The use of a criterion function restricts the beliefs of econometricians about elasticities from a large set to a single point. Mountford and Uhlig (2009) select the non-policy shock that explains as much movement as possible in output, tax revenue, consumption, and non-residential investment, i.e. in the variables subject to sign restrictions. The output elasticity of tax revenue selected by the penalty function is 3.32. The resulting impact tax multiplier is 0.46. The output elasticity of government spending selected by the penalty function is 0.20. The resulting

\(^{23}\)This discussion of the penalty function approach is non-technical. For a detailed analysis of the analytics of the penalty function approach see Caldara and Kamps (2010).

\(^{24}\)The criterion function solves a constrained optimization problem. The numerical procedure finds the impulse vector that maximizes the forecast error variance of some variables in the VAR, subject to sign restrictions. The criterion function constructed by Mountford and Uhlig (2009) penalizes impulse vectors that violate sign restrictions. This is why this approach is also known as the penalty function approach.
impact spending multiplier is 0.35. The probability of the spending multiplier to be larger than the tax multiplier is 0.22, as reported in Table 4.

Of course I could set up a different penalty function. One of the infinitely possible penalty functions could select the non-policy shock that explains as much variability as possible in all variables in the VAR. As shown in Caldara and Kamps (2010), the specification of the penalty function has a large impact on results. At the same time, it seems hard to impose discipline in the choice of penalty functions.

Summing up, thanks to the analytical characterization of the relation between output elasticities and fiscal multipliers, I study in some depth the implications of existing identification schemes. I have shown how the schemes used in the literature select different restrictions on the output elasticities of tax revenue and government spending. The analysis of four different identification schemes produces different estimates of impact fiscal multipliers. The recursive approach and traditional SVAR analysis suggests that spending multipliers are larger than tax multipliers. The sign restriction approach produces a wide range of multipliers, and it does not provide a clear cut answer. Finally, the sign restriction approach with penalty function suggests that impact tax multipliers are larger than impact spending multipliers.

3.5 Deriving Restrictions on Elasticities

What should we make of the uncertainty around output elasticities of fiscal variables found in the previous section? Some of the identification schemes appear very dogmatic, selecting a single value of the relevant output elasticity. Others appear quite loose, imposing almost no restrictions on the relevant elasticity. In this section, I attempt to strike a balance between these two extremes, by imposing prior distributions on the output elasticities of fiscal variables. First, I show how to derive prior distributions on elasticities from disaggregated data on tax and spending categories, and from micro-
econometric studies. Second, I show how to derive distributions on elasticities from a simple DSGE model.

### 3.5.1 Priors on Elasticities from Disaggregated Data

The OECD\textsuperscript{25} estimates the output elasticity of tax revenue for four different tax categories: personal income tax, social security contributions, corporate income tax, and indirect taxes. In addition, the OECD estimates the output elasticity of transfers which, following B&P, I treat as negative taxes. B&P aggregate these elasticities to obtain a point estimate for the output elasticity $\eta_{T,Y}$ according to the following aggregator:

$$\eta_{T,Y} = \sum_{i} \eta_{T_i,Y} \frac{T_i}{T},$$

(3.7)

where $i$ denotes the tax category, $T_i$ denotes the level of tax revenue, $T$ denotes total tax revenue, and $\eta_{T_i,Y}$ denotes the elasticity of taxes of type $i$ to output. Following B&P, I evaluate $T_i$ and $T$ at their sample mean.

The elasticity $\eta_{T_i,Y}$ can be separated into two components: the elasticity of tax revenue with respect to the relevant tax base, $\eta_{T_i,TB_i}$, and the elasticity of the tax base relative to output $\eta_{TB_i,Y}$:

$$\eta_{T_i,Y} = \eta_{T_i,TB_i} \eta_{TB_i,Y}.$$  

(3.8)

The OECD computes elasticities of tax revenue with respect to their base $\eta_{T_i,TB_i}$ from legislation and micro data. Table 8 reports the elasticities constructed by the OECD for the United States.

The OECD estimates elasticities of tax bases with respect to output $\eta_{TB_i,Y}$ using linear regressions. B&P only use point estimates from OLS regressions and do not include a measure of uncertainty. Instead, I estimate

\textsuperscript{25}Giorno et al. (1995); Girouard and André (2005); van den Noord (2000)
the output elasticities of tax bases using Bayesian linear regressions.\textsuperscript{26} In the Bayesian framework regression coefficients are random variables. Table 8 reports the median and the 68\% credible set for the sub-elasticities.\textsuperscript{27} The first row of Table 5 reports the median and the 68\% credible set for the aggregate output elasticity of tax revenue $\eta_{T,Y}$, which I construct aggregating sub-elasticities according to equations (3.7) and (3.8). Sampling uncertainty is small. The median value for $\eta_{T,Y}$ is 2.26, while the 68\% credible set ranges between 2.16 and 2.37. Little sampling uncertainty translates into sharp predictions for the tax multiplier, which does not differ substantially from the estimates obtained adopting the B&P approach.

Small sampling uncertainty does not necessarily mean that there is little uncertainty around $\eta_{T,Y}$. The construction of $\eta_{T,Y}$ is based on a large number of assumptions and the econometrician (i.e., the OECD) is in control of a large number of details (e.g., definition of variables and data sources) that can have a big impact on the estimates. Let us consider two important aspects.

First, the OECD constructs the elasticity of personal income taxes and social security contributions for 22 industrialized countries, including the U.S. At least for the U.S. there are alternative data sources. For instance, the NBER estimates the output elasticity of personal income taxes and social security contributions using the TAXSIM model (Feenberg and Coutts, 1993). This model implements a micro-simulation of the U.S. federal and state income tax systems. The model is based on a large sample of actual tax returns prepared by the Statistics of Income Division of the Internal Revenue Service. Figure 4 plots the values of the elasticity as computed by the OECD, and as computed using the TAXSIM model.\textsuperscript{28} The NBER computes the output elasticity of tax revenue annually from 1960, while the OECD computes the

\textsuperscript{26}Refer to Appendix A.3 for details.

\textsuperscript{27}The Bayesian set estimates are called credible sets. This is analogous to the concept of confidence intervals used in classical statistics.

\textsuperscript{28}The OECD measure excludes social security contributions. Including them makes little difference for the comparison.
elastici

ty every fifth year from 1979. B&P computes the average elasticity and apply it for the entire sample starting 1947. Hence, having a longer
time series is important. The OECD estimates also display large fluctua-
tions, which seem at odd with the recent history of the US tax system. In
particular, the elasticity drops threefold from 3.9 from 1992 to 1.3 in 1996.
The largest fiscal bill passed during these years is OBRA 1993, signed by
President Clinton, which increased (and not decreased) the progressivity of
the tax system (CBO, 1994). Differences between the two series is large: the
average elasticity of personal income taxes and social security contributions
estimated using TAXSIM is 1.65, while the average OECD elasticity is 2.54.30

Second, the OECD assumes that the elasticity of private consumption to
output is 1. When I estimate this elasticity using a Bayesian linear regression,
I obtain a median value of 0.55. An elasticity lower than one seems in line with
micro studies estimating the response of consumption to temporary income
shocks. For instance, Johnson et al. (2006) show that households receiving a
lump-sum tax rebate spend on consumption of non-durable goods between
20% and 40% of the rebate within one quarter.31

The second row of Table 5 reports the median and the 68% credible set
for the output elasticity of tax revenue estimated using TAXSIM data and
an estimated output elasticity of private consumption. The median elasticity
is 1.81, and again there is little sampling uncertainty. At the same time the
median impact multiplier declines by 68%, dropping from 0.19 for a median

---

29The only exception is for the years 1992, 1993.
30I am currently working on including in my calculations estimates of the output elastici

ty of fiscal variables obtained by Cohen and Follette (2000) and Follette and Lutz (2010). The inclusion of these elasticities is important because the CBO adopts a similar
estimation methodology (CBO, 2010a).
31See also Souleles (1999), Souleles (2002), Shapiro and Slemrod (2003b), Shapiro and
Slemrod (2003a). I made the following back of the envelope calculation. Assume that a
household has an income of 60000$ and consumes 70% of her income. A 600$ tax rebate (as
in 2001) increases income by 1%. If the household spends 40% of the rebate on consumption
goods, consumption increases by 0.57%, in line with the reduced-form estimates of the
output elasticity of consumption.
3.5. DERIVING RESTRICTIONS ON ELASTICITIES

elasticity $\eta_{T,Y} = 2.26$, to 0.06, for a median elasticity $\eta_{T,Y} = 1.81$.

Most authors in the VAR literature assume that the output elasticity of government consumption and investment $\eta_{G,Y}$ is zero. The OECD also makes this assumption and does not provide estimates for the elasticity. There seems to be consensus that government consumption and investment do not have components that react automatically to economic fluctuations. Recent studies challenge this assumption. For instance, Rodden and Wibbels (2010) find evidence of pro-cyclical fiscal policy among state and local governments in the United States. In particular, for the period 1977 to 1997 they find an elasticity of state and local spending to output equal to 0.17. Lamo et al. (2007) also estimate pro-cyclical components of government consumption. Lane (2003) finds that public wages are pro-cyclical in most OECD economies. These studies are based on annual data, and pro-cyclicality might disappear at quarterly frequency. Despite this shortcoming, the possibility that at least part of government spending is weakly pro-cyclical cannot be discarded a-priori. For this reason, I construct a prior on the output elasticity of spending that takes value zero with probability 0.5, and it is normally distributed with mean 0.17 with probability 0.5. I show the associated spending multiplier in Table 6. The median spending multiplier is 0.54, and the 68% credible set ranges between 0.33 and 0.75. The spending multiplier is well below 1.

Summing up, this section shows how prior distributions on the output elasticities of tax revenue and government spending can be computed from micro data and legislation. I have shown that sampling uncertainty is small, while the use of alternative source of data can shift the distributions. What are the implications for the relative size of impact spending and tax multipliers? Table 7 reports the probability that the spending multiplier is larger than the tax multiplier. If I construct the output elasticity of tax revenue fol-

---

32 The authors use OLS panel regressions. The coefficient is significant at 1% level.
33 Gomes (2009) finds that in a model with search frictions pro-cyclical public wages are optimal. He also finds that public wages have been weakly pro-cyclical in the United States.
lowing the OECD / B&P methodology, this probability is 0.94. If I integrate the B&P analysis using elasticities from the TAXSIM model and estimating the output elasticity of consumption the probability jumps to 1. I conclude that the evidence suggests that impact spending multipliers are larger than impact tax multipliers.

3.5.2 Priors on Elasticities from a DSGE Model

In this section, I show how to derive a prior distribution for the output elasticities of tax revenue and government spending from DSGE models. To achieve this goal, I only specify the profit maximization problem of the firm and the fiscal sector.

I assume that the model is populated by a continuum of firms operating in a competitive market. They produce goods $y_t$ using a Cobb-Douglas production function:

$$y_t = z_t k_t^\alpha l_t^{1-\alpha}.$$

where $y_t$ is output, $k_t$ is the capital stock, $l_t$ is labor, $\alpha$ is the capital income share in production, and $z_t$ is an exogenous technology process, that for convenience I assume to be stationary. The wage $w_t$ paid by the firm to hire labor and the return $r_t$ paid on capital are determined on competitive markets and are derived from the first-order conditions of the firm’s profit maximization problem:

$$r_t = \alpha z_t k_t^{\alpha-1} l_t^{1-\alpha},$$

$$w_t = (1 - \alpha) z_t k_t^\alpha l_t^{-\alpha}.$$

If the production function is Cobb-Douglas, capital income and labor income are constant shares of output:

$$r_t k_t = \alpha y_t \quad \text{(3.9)}$$

$$w_t l_t = (1 - \alpha) y_t \quad \text{(3.10)}$$
3.5. DERIVING RESTRICTIONS ON ELASTICITIES

Assume that a fiscal authority levies taxes on labor income and capital income according to the following progressive tax schedules:

$$
\tau^k_t = t_{0,k} \left(r^t_k \right)^{t_{1,k}} e^{\epsilon^t_k},
\tau^l_t = t_{0,l} \left(w^t_l \right)^{t_{1,l}} e^{\epsilon^t_l},
$$

where $\tau^k_t$ is the tax rate on capital income, $\tau^l_t$ is the tax rate on labor income, where $t_{0,i}$ denotes the average tax rate and $t_{1,i}$ the degree of non-linearity.\(^{34}\)

The government sets spending $g_t$ according to the following rule:

$$
g_t = y_t^{1-\varphi_g} e^{\epsilon^t_g}.
$$

The fiscal authority runs a balanced-budget in every period:

$$
tr_t = g_t - T_t,
$$

(3.11)

where $tr_t$ denotes transfers to the households and $T_t$ denotes tax revenue:

$$
T_t = \tau_{t,wt} w^t_l + \tau_{t,rt} r^t_k.
$$

(3.12)

Substituting (3.9) and (3.10) into (3.12) and in the fiscal rule I obtain:

$$
\begin{align*}
T_t &= \tau^l_t (1 - \alpha) y_t + \tau^k_t \alpha y_t \\
\tau^l_t &= t_{0,l} [ (1 - \alpha) y_t ]^{t_{1,l}} \exp (\epsilon^t_{l,t}) \\
\tau^k_t &= t_{0,k} [ \alpha y_t ]^{t_{1,k}} \exp (\epsilon^t_{k,t})
\end{align*}
$$

If I log-linearize these three expressions around the steady state, I can write aggregate tax revenue as:

$$
\hat{T}_t = \frac{\tau^k_t \alpha}{\tau^k_t \alpha + \tau^l_t (1 - \alpha)} [(1 + \varphi_k) \hat{y}_t + \epsilon^t_{k,t}] + \frac{\tau^l_t (1 - \alpha)}{\tau^k_t \alpha + \tau^l_t (1 - \alpha)} [(1 + \varphi_l) \hat{y}_t + \epsilon^t_{l,t}],
$$

\(^{34}\)Results hold if I introduce an autoregressive (stationary) component.
where $\varphi_k = t_{0,k}t_{1,k}$, $\varphi_l = t_{0,l}t_{1,l}$, $\hat{x}$ denotes percentage deviations from steady state, and variables without time index are steady state values. The output elasticity of tax revenue is the percentage response of tax revenue to non-policy shocks (e.g., technology shocks) that increases output by 1%:

$$\eta_{T,Y}^{DSGE} = \frac{\tau_k \alpha}{\tau_k \alpha + \tau_l (1 - \alpha)} [(1 + \varphi_k)] + \frac{\tau_l (1 - \alpha)}{\tau_k \alpha + \tau_l (1 - \alpha)} [(1 + \varphi_l)].$$

If I log-linearize the spending rule I obtain:

$$\hat{g}_t = -\varphi_g \hat{y}_t + \epsilon_{t,g}.$$ 

The output elasticity of government spending is the percentage response of government spending to non-policy shocks (e.g., technology shocks) that increases output by 1%:

$$\eta_{G,Y}^{DSGE} = -\varphi_g.$$

In order to derive a distribution on the elasticities, I need to calibrate and/or impose distributions over the structural parameters of the DSGE. In this simple example, I set parameters following the recent work by Leeper et al. (2010). They calibrate $\alpha$ to 0.3, which implies a labor share of 70%. The steady-state tax rates are 0.223 for the labor income tax and 0.184 for the capital income tax. Leeper et al. (2010) assume a gamma prior distribution for the parameters of the fiscal rules.\(^{35}\) Table 9 summarizes the calibration and provides details of the distributions. As mentioned in Leeper et al. (2010): “Priors for the fiscal parameters were chosen to be fairly diffuse and cover a reasonably large range of parameter values”. The idea of the exercise is exactly to use DSGE models to summarize knowledge about the U.S. tax and spending system by prominent economists in the literature to

\(^{35}\)Alternatively I could have used the posterior distributions estimated by Leeper et al. (2010). As shown in their paper, the data does not seem to be informative about these parameters and posterior distributions are close to the priors.
form distributions on the output elasticities in the SVAR.36

The last row of Table 5 reports the distribution of the output elasticity of tax revenue. The median elasticity is 1.62, and the 68% credible set ranges between 1.45 and 1.92. This elasticity is slightly lower than the elasticity I calculated from micro data and legislation.

When I account for sampling uncertainty, the associated multiplier is not different from zero. The last row of Table 6 reports the output elasticity of government spending. Leeper et al. (2010) assume that spending is weakly counter-cyclical. A negative elasticity implies a larger spending multiplier (0.79) compared to what I computed using evidence from micro studies. The probability that the spending multiplier is larger than the tax multiplier, reported in Table 7, is 1.

I have proposed two alternative methods to compute prior distributions for the output elasticities of tax revenue and government spending. The first method relies on the use of micro data and legislation, while the second derives prior distributions for elasticities using a simple DSGE model. The outcome of both analysis is similar: it is extremely likely that the impact spending multiplier is larger than the tax multiplier.

3.6 Extensions

In this section, I analyze fiscal multipliers at longer horizons. I then analyze the response of consumption, investment and the real wage to fiscal shocks.

36Non-linear fiscal rules are also used in an estimated open economy model for the Euro Area written by the European Commission (Ratto et al. (2006)). The authors calibrate the parameters of the fiscal rules using the OECD data on the elasticities described in the previous section. For this reason I did not pursue this alternative calibration strategy.
3.6.1 Dynamic Multipliers

The analytical framework presented in Section 3.3 can be used to study impulse responses at longer horizon than one period. The Moving Average (MA) representation of the SVAR model is:

\[ y_t = \sum_{j=0}^{\infty} \Theta_j e_{t-j}, \]  

(3.13)

where \( \Theta_j = \Phi_j A^{-1} \) \((j = 0, 1, 2, \ldots)\). The elements of matrices \( \Phi_j \)'s are functions of the autoregressive coefficients contained in the lag polynomial \( B(L) \). The matrix \( \Theta_j \) contains impulse responses \( j \) quarters after the shock.

I do not attempt to inspect the analytical expression for impulse responses at horizon \( j \geq 1 \). I evaluate \( \Theta_j \) and \( \Sigma_u \) at the OLS estimates. Assumption 1 ensures that, for given \( \Theta_j \) and \( \Sigma_u \), impulse responses to a policy shock \( e_{P,t} \) only depend on the output elasticity of the policy variable at any horizon. I can plot impulse responses as non-linear functions of the elasticities for any value of the elasticities, as I did for the impact responses.\(^{38}\)

Figure 5 plots the tax multiplier 4, 8, and 12 quarters after the shock. The tax multiplier 4 quarters after the shock is zero for an output elasticity of tax revenue close to 1.5. For a given positive elasticity, tax multipliers increase with the horizon.

Figure 6 plots the spending multiplier 4, 8, and 12 quarters after the shock. The spending shocks seem to have a persistent effect on output, as spending multipliers decline over time very slowly.

Figure 7 plots tax and spending multipliers estimated using different priors on the output elasticities of fiscal variables. Tax multipliers are close to

\(^{37}\)Impact responses can be written as \( y_t = \Theta_0 A^{-1} e_t \). Since \( \Phi_0 = I \), I obtain the expressions studied in section 3.3.

\(^{38}\)If I did not have analytical expressions, I should have solved a system of 55 non-linear equations for each value of the elasticity I wanted to study, and then compute impulse responses at different horizons. The analytical procedure is substantially faster and possibly more accurate.
zero for up to 8 quarters after the tax cut. The peak multiplier is 0.4 and it is recorded 12 quarters after the policy intervention. There is large uncertainty surrounding spending multipliers at longer horizons. Yet, irrespective of the prior distribution on the output elasticity of governmen spending, the median spending multiplier is above 0.5, and a zero multiplier is excluded by the 68% credible set at least for 7 quarters after the policy intervention.

I compute the probability that the spending multiplier is larger than the tax multiplier up to 16 quarters after the policy interventions, for the prior distributions computed in Section 3.5. Results are reported in Figure 8. When I derive priors on elasticities from DSGE models, this probability is above 85% up to 16 quarters after the policy interventions. The probability is well above 80% when I compute priors distribution from the data. From this analysis I conclude that spending multipliers are likely to be larger than tax multipliers up to four years after the policy interventions.

3.6.2 Output Components

Figure 9 plots the OLS estimate for the response of consumption, non-residential investment, and real wages to a spending shock (left column) and to a tax shock (right column). Table 10 reports the median and the 68% credible set for the response of output components when I compute prior distributions of elasticities from the data as shown in Section 3.5.1.

The top-left panel of Figure 8 plots the response of consumption to a spending shock. This response is negative for $\eta_{G,Y} \geq 0$. Table 10 confirms that these responses are negative also when I add sampling uncertainty. This response is in line with the evidence documented by Ramey and Shapiro (1998). Mountford and Uhlig (2009) find that the response of consumption to a spending shock is not different from zero. B&P instead find that this response is positive. A number of papers has attempted to rationalize the positive response of consumption to a spending shocks in New Keynesian models (Gali et al., 2007). Evidence is instead in favor of standard New-
Keynesian and neoclassical models, which predict a decline in consumption due to the negative wealth effect associated to an increase in public spending.

The response of non-residential investment to a spending shock is negative for $\eta_{G,Y} \geq 0$. The response of real wages to a spending increase is negative for all elasticities larger than $-1$. When I compute the distribution for elasticities from the data, the response is significantly negative. I report results for real wages as Neoclassical and New Keynesian models tend to predict different signs for the responses of this variable to spending shocks. The former class of models predicts a negative response, while the latter a positive response (Ramey and Shapiro, 1998). The evidence presented in this section seems to be consistent with neoclassical models.

The response of consumption and investment to a tax cut depends on the prior distribution of $\eta_{T,Y}$. When I compute the distribution for $\eta_{T,Y}$ from the data (a median elasticity of 1.81) the response of consumption and investment are zero. The response of real wages is negative for all positive output elasticities of tax revenue.

3.7 Conclusions

In this paper I study fiscal multipliers for the U.S. based on data for the period 1947-2010. I show that spending multipliers have been larger than tax multipliers up to four years after policy interventions. To reach this conclusion, I compare analytically different identification schemes used in the SVAR literature. I show that differences in estimates of fiscal multipliers documented in the literature by Blanchard and Perotti (2002) and Mountford and Uhlig (2009) are mostly due to different restrictions on the output elasticities of tax revenue and government spending. I then proposed two departures from the existing literature, computing distributions of elasticities using different empirical methodologies, and a class of DSGE models.

The analytical framework developed in this paper can be applied to study
identification problems in a large class of time-series models, such as VARs with time-varying reduced-form coefficients and regime-switching VARs. The first class of models has been recently used by Cimadomo et al. (2010) to study whether fiscal multipliers change over time in the Euro Area. The latter class of models has been used by Auerbach and Gorodnichenko (2010) to study whether fiscal multipliers depend on the state of the economy. The analytical framework proposed in this paper can help to unveil whether the relation between output elasticities and fiscal multipliers has changed over time and whether time variation in elasticities can be used to better identify fiscal shocks.

The comparative framework can also be applied to time-series models that are better suited to deal with fiscal foresight. Forni and Gambetti (2010) estimate a factor model to estimate truly unanticipated shocks. They identify spending shocks imposing contemporaneous and sign restrictions on the VAR representation of the factors. Mertens and Ravn (2010) construct an SVAR estimator to identify anticipated fiscal shocks. They also provide inference on the effects of unanticipated shocks that are identified using standard identification schemes. The identification problem in this class of models can also be analyzed within the framework described in this paper.

Bibliography


THE ANALYTICS OF SVARS


Lane, P. R., “The cyclical behaviour of fiscal policy: evidence from the OECD,” Journal of Public Economics 87 (December 2003), 2661–2675.


Lütkepohl, H., New introduction to multiple time series analysis (Springer, 2005).


UHLIG, H., “What are the effects of monetary policy on output? Results from an agnostic identification procedure,” *Journal of Monetary Economics* 52 (March 2005), 381–419.


A. Appendix

A.1 Analytics

The reduced-form VAR model is

\[ X_t = \mu + B(L) X_{t-1} + u_t, \tag{3.14} \]

where \( X_t \) is a \( n \times 1 \) vector of endogenous variables, \( \mu \) is a constant, \( B(L) \) is a lag polynomial of order \( L \), and \( u_t \) is an \( n \times 1 \) vector of one-step-ahead prediction errors. I assume a Gaussian distribution of \( u_t \) with mean zero and symmetric covariance matrix \( E[u_t u'_t] = \Sigma_u = [\sigma_{ij}] \). I also assume that the variance covariance matrix \( \Sigma_u \) is positive definite. I denote the standard deviation of prediction error \( u_{i,t} \) as \( \sigma_i \), with \( \sigma_i \equiv \sqrt{\sigma_{ii}}, \ i = 1,...,n_x \). I further denote the correlation coefficient between errors \( u_{i,t} \) and \( u_{j,t} \) as \( \rho_{ij} \), with \( \rho_{ij} \equiv \sigma_{ij} / (\sigma_i \sigma_j) \) \( i, j = 1,...,n_x \).

The relation between reduced-form residuals \( u_t \) and structural shocks \( e_t \) is given by:

\[ Au_t = e_t, \]
where $A$ is an $(n \times n)$ matrix of structural coefficients. I assume a Gaussian distribution of $e_t$ with mean zero and diagonal covariance matrix $\Sigma_e$.

Pre-multiplying equation (3.14) by matrix $A$ gives the structural form of the VAR model:

$$AX_t = AB(L)X_{t-1} + e_t.$$ 

Finally, the relation between structural coefficients $(A, \Sigma_e)$ and reduced-form coefficients $\Sigma_u$ is given by:

$$E[u_t u_t'] = E[A^{-1} e_t e_t' A^{-1}'],$$

$$E[u_t u_t'] = A^{-1} E[e_t e_t'] A^{-1},$$

$$\Sigma_u = A^{-1} \Sigma_e A^{-1'},$$ \hspace{1cm} (3.15)

which describes a system of $n(n-1)/2$ independent non-linear equations.

**Bivariate Models**

In the bivariate model I solve a system of three equations (as many as the distinct elements of $\Sigma_u$) in three unknowns $(a_{Y,P}, \epsilon_{YY}, \epsilon_{PP})$:

$$\Sigma_u = A^{-1} \Sigma_e A^{-1'},$$

where

$$\Sigma_u = \begin{bmatrix} \sigma_{YY} & \sigma_{YP} \\ \sigma_{YP} & \sigma_{PP} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 1 & -a_{Y,P} \\ -\eta_{P,Y} & 1 \end{bmatrix},$$

$$\Sigma_e = \begin{bmatrix} \epsilon_{YY} & 0 \\ 0 & \epsilon_{PP} \end{bmatrix}.$$
A. APPENDIX

The solution of this system is:

\[ a_{Y,P} = \frac{\eta_{P,Y}\sigma_{YY} - \sigma_{YP}}{\eta_{P,Y}\sigma_{YP} - \sigma_{PP}} \]

\[ \epsilon_{YY} = \frac{\left( \sigma_{PP} + \eta_{P,Y}^2\sigma_{YY} - 2\eta_{P,Y}\sigma_{YP}\right) \left( \sigma_{PP}\sigma_{YY} - \sigma_{YP}^2 \right)}{\left( \sigma_{PP} - \eta_{P,Y}\sigma_{YP} \right)^2} \]

\[ \epsilon_{PP} = \sigma_{PP} + \eta_{P,Y}^2\sigma_{YY} - 2\eta_{P,Y}\sigma_{YP}. \]

Substituting the analytical solution for \( a_{Y,P} \) in matrix \( A^{-1} \) I obtain the following analytical expression for the impact impulse responses:

\[ A^{-1}(\eta_{P,Y}, \Sigma_u) = \frac{\sigma_{PP} - \eta_{P,Y}\sigma_{YP}}{\eta_{P,Y}\sigma_{YY} + \sigma_{PP} - 2\eta_{P,Y}\sigma_{YP}} \left[ \begin{array}{c} 1 \\ \eta_{P,Y} \frac{\sigma_{YP} - \eta_{P,Y}\sigma_{YY}}{\sigma_{PP} - \eta_{P,Y}\sigma_{YP}} \end{array} \right]. \]

The assumption that \( \Sigma_u \) is positive definite ensures that the denominator of all impact responses is strictly larger than zero. This guarantees that impulse response functions are defined for all output elasticities \( \eta_{P,Y} \).\(^{39}\)

**Multivariate Models**

In the bivariate model I can rewrite the element \( i \) of the impulse vector associated to the policy shock \( e_{2,t} \) as:

\[ A_{i,P}^{-1} = \frac{\sigma_{iP} - \eta_{P,Y}\sigma_{iY}}{\sigma_{PP} + \eta_{P,Y}^2\sigma_{YY} - 2\eta_{P,Y}\sigma_{YP}}, \]  

(3.16)

for \( i = Y, P. \)

\(^{39}\)Let \( a \) be a \( 2 \times 1 \) vector. The function \( a\Sigma_a a' \) is called a quadratic form in \( a \). Matrix \( \Sigma_u \) is positive definite if \( a\Sigma_u a' > 0 \) for all \( a \neq 0 \). For \( a = [\eta_{P,Y}, 1] \) I can write this condition as \( \eta_{P,Y}^2\sigma_{YY} + \sigma_{PP} - 2\eta_{P,Y}\sigma_{YP} > 0 \). See Golub and van Loan (1996)
Let us introduce a third variable to the system:

\[
\begin{align*}
    u_{Y,t} &= a_{Y,P}u_{P,t} + a_{Y,3}u_{3,t} + e_{Y,t} \\
    u_{P,t} &= \eta_{P,Y}u_{Y,t} + a_{P,3}u_{3,t} + e_{P,t} \\
    u_{3,t} &= a_{3,Y}u_{Y,t} + a_{3,P}u_{P,t} + e_{3,t}.
\end{align*}
\]

In a 3-equation VAR model I need three restrictions to identify system (3.15). Without loss of generality I assume that restrictions are imposed on \(\eta_{P,Y}, a_{P,3},\) and \(a_{Y,3}\). In the interest of space I do not report the solution to the 3-equation system.\(^{40}\) The element \(i\) of the impulse vector associated to the policy shock \(e_{P,t}\) can be written as:

\[
A^{-1}_{i,P} = \frac{\sigma_{iP} - \eta_{P,Y}\sigma_{iY} - a_{P,3}\sigma_{i3}}{\sigma_{PP} + \eta_{P,Y}\sigma_{YY} + a_{P,3}\sigma_{33} - \eta_{P,Y}\sigma_{YP} - 2a_{P,3}\sigma_{P3} + 2\eta_{P,Y}a_{P,3}\sigma_{Y3}}.
\]

Notice that the impulse vector is independent of the restriction on \(a_{13}\) that I need to impose to identify \(e_{Y,t}\).

If \(a_{P,3} = 0\), the solution for the impulse vector (3.17) collapses to expression (3.16), the solution for the impulse vector found in the bivariate model. This result generalizes to VAR models dimension \(n_x\). If I restrict the response of the policy variable \(u_{P,t}\) to react only to unexpected changes in output \(u_{Y,t}\), expression (3.16) describes the impact response of variable \(i\) to a policy shock, for \(i = Y, P, ..., n\). These restrictions are imposed by Assumption 2 in the main text.

Furthermore, if \(a_{Y,3} = 0\), the element \(i\) of the impulse vector associated to the policy shock \(e_{1,t}\) can be written as:

\[
A^{-1}_{i,Y} = \frac{(\sigma_{PP} - \eta_{P,Y}\sigma_{YP})(-\eta_{P,Y}\sigma_{YP}\sigma_{Yi} + \sigma_{Yi}\sigma_{PP} + \eta_{P,Y}\sigma_{YY}\sigma_{P3} - \sigma_{YP}\sigma_{P3})}{(\eta_{P,Y}\sigma_{YY} - 2\eta_{P,Y}\sigma_{YP} + \sigma_{PP})(-\sigma_{YP}^2 + \sigma_{YY}\sigma_{PP})},
\]

for \(i = Y, P, 3\). In a system of dimension \(n_x\) under Assumption 1, \(a_{1i} = 0\),

\(^{40}\)Results are available on request.
A. APPENDIX

for \( i = 3, \ldots, n_x \). Expression (3.18) describes the response of variable \( i \), for \( i = 1, \ldots, n \).

A.2 Proofs

Proof of Proposition 1. The impact response of output to a policy shock is:

\[
A_{Y,P}^{-1} (\eta_{P,Y}, \Sigma_u) = \frac{\sigma_{YP} - \eta_{P,Y} \sigma_{YY}}{\eta_{P,Y}^2 \sigma_{YY} + \sigma_{PP} - 2 \eta_{P,Y} \sigma_{YP}}.
\]

First, I prove existence of a global minimum and maximum of \( A_{Y,P}^{-1} (\eta_{P,Y}, \Sigma_u) \). Note that \( A_{Y,P}^{-1} (\eta_{P,Y}, \Sigma_u) \) belongs to the family of rational functions, which are continuous and differentiable. So in order to find the global extrema of \( A_{Y,P}^{-1} (\eta_{P,Y}, \Sigma_u) \) I have to investigate its first and second derivatives. With some abuse of notation, denote \( A_{Y,P}^{-1} (\eta_{P,Y}, \Sigma_u) \) by \( f (\eta_{P,Y}) \). The first derivative of \( f (\eta_{P,Y}) \) is:

\[
f'(\eta_{P,Y}) = \frac{\eta_{P,Y}^2 \sigma_{YY}^2 - 2 \eta_{P,Y} \sigma_{YY} \sigma_{YP} + 2 \sigma_{YP} - \sigma_{YY} \sigma_{PP}}{(\eta_{P,Y}^2 \sigma_{YY} - 2 \eta_{P,Y} \sigma_{YP} + \sigma_{PP})^2}.
\]

Equating the derivative to zero I obtain two points that satisfy the necessary conditions for an extremum of \( f (\eta_{P,Y}) \):

\[
\eta_{P,Y}^{\text{min}} = \frac{\rho_{YP} + \sigma_P \sqrt{1 - \rho_P^2}}{\sigma_Y}, \quad \eta_{P,Y}^{\text{max}} = \frac{\rho_{YP} - \sigma_P \sqrt{1 - \rho_P^2}}{\sigma_Y}.
\]

It is immediate to see that \( \eta_{P,Y}^{\text{min}} > \eta_{P,Y}^{\text{max}} \). The sufficient condition for extremum is checked deriving the second derivatives of \( f (\eta_{P,Y}) \) and evaluating it at \( \eta_{P,Y}^{\text{min}} \) and \( \eta_{P,Y}^{\text{max}} \):

\[
f'' \big( \eta_{P,Y} \big) \big|_{\eta_{P,Y}=\eta_{P,Y}^{\text{min}}} = \frac{\sigma_Y^3 \sqrt{1 - \rho_{PY}^2}}{2 \sigma_P^3} > 0,
\]

\[
f'' \big( \eta_{P,Y} \big) \big|_{\eta_{P,Y}=\eta_{P,Y}^{\text{max}}} = -\frac{\sigma_Y^3 \sqrt{1 - \rho_{PY}^2}}{2 \sigma_P^3} < 0,
\]
provided that $|\rho_{PY}| < 1$.

Finally, the global minimum and maximum of $A_{Y,P}^{-1}(\eta_{P,Y}, \Sigma_u)$ are:

$$A_{Y,P}^{-1}(\eta_{P,Y}^{\text{min}}, \Sigma_u) = -\frac{\sigma_Y}{2\sigma_P \sqrt{1 - \rho_{YP}^2}} < 0$$

$$A_{Y,P}^{-1}(\eta_{P,Y}^{\text{max}}, \Sigma_u) = \frac{\sigma_Y}{2\sigma_P \sqrt{1 - \rho_{YP}^2}} > 0.$$

The second statement in Proposition 1 can be easily proved using the definition of $A_{Y,P}^{-1}(\eta_{P,Y}, \Sigma_u)$:

$$A_{Y,P}^{-1}(\eta_{P,Y}, \Sigma_u) = 0 \iff \sigma_Y - \eta_{P,Y} \sigma_{YY} = 0 \iff \eta_{P,Y} = \frac{\sigma_{YY}}{\sigma_Y}.$$

The third statement in Proposition 1 states that $A_{Y,P}^{-1}(\eta_{P,Y}, \Sigma_u)$ is strictly decreasing for $\eta_{P,Y} \in [\eta_{P,Y}^{\text{max}}, \eta_{P,Y}^{\text{min}}]$ and strictly increasing for $\eta_{P,Y} < \eta_{P,Y}^{\text{max}} \vee \eta_{P,Y} > \eta_{P,Y}^{\text{min}}$. This statement can be easily proved by analyzing the sign of $f'(\eta_{P,Y})$. ■

**Proof of Proposition 2.** The impact response of output ($Y_t$) to a non-policy shock is given by:

$$A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) = \frac{\sigma_{TT} - \eta_{T,Y} \sigma_{YT}}{\sigma_{TT} - 2\eta_{T,Y} \sigma_{YT} + \eta_{T,Y}^2 \sigma_{YY}}.$$

The impact response of tax revenue ($T_t$) to a non-policy shock is given by:

$$A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) = \frac{\eta_{T,Y}(\sigma_{TT} - \eta_{T,Y} \sigma_{YT})}{\sigma_{TT} - 2\eta_{T,Y} \sigma_{YT} + \eta_{T,Y}^2 \sigma_{YY}}.$$

First notice that sign restrictions imposed in Proposition 2 are satisfied by non-policy shocks that generate either a positive response of both output and tax revenue, or a negative response of both output and tax revenue. In the latter case the candidate shock is a negative shock. Multiplying the associated impulse vector by $-1$ I obtain a positive shock that satisfies the
sign restriction.

Hence I have to find the set of $\eta_{T,Y}$ such that:

\[ A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0 \land A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0, \]

or

\[ A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0 \land A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0. \]

I first examine positive non-policy shocks. Define $S_{Y,Y}^+$ as the set of $\eta_{T,Y}$ for which $A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0$ and $S_{T,Y}^+$ as the set of $\eta_{T,Y}$ for which $A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0$. Using Assumption 2 I find that:

\[ S_{Y,Y}^+ = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} < \frac{\sigma_{TT}}{\sigma_{YT}} \right\}, \quad (3.19) \]

\[ S_{T,Y}^+ = \left\{ \eta_{T,Y} \in \mathbb{R} : 0 < \eta_{T,Y} < \frac{\sigma_{TT}}{\sigma_{YT}} \right\}. \quad (3.20) \]

The intersection of $S_{Y,Y}^+$ and $S_{T,Y}^+$ represents all $\eta_{T,Y}$ for which $A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0 \land A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0$:

\[ S_{Y,Y}^+ \cap S_{T,Y}^+ = \left\{ \eta_{T,Y} \in \mathbb{R} : 0 < \eta_{T,Y} < \frac{\sigma_{TT}}{\sigma_{YT}} \right\}. \]

Similarly, define $S_{Y,Y}^-$ as the set of $\eta_{T,Y}$ for which $A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0$ and $S_{T,Y}^-$ as the set of $\eta_{T,Y}$ for which $A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0$. Using Assumptions 1 and 2 I find that:

\[ S_{Y,Y}^- = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} > \frac{\sigma_{TT}}{\sigma_{YT}} \right\}, \quad (3.21) \]

\[ S_{T,Y}^- = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} < 0 \lor \eta_{T,Y} > \frac{\sigma_{TT}}{\sigma_{YT}} \right\}. \quad (3.22) \]

The intersection of $S_{Y,Y}^-$ and $S_{T,Y}^-$ represents all $\eta_{T,Y}$ for which $A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0$.

A. APPENDIX

111
0 \wedge A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0:

\[
S_{Y,Y}^- \cap S_{T,Y}^- = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} > \frac{\sigma_{TT}}{\sigma_{YT}} \right\}.
\]

Finally, the union of the intersections \((S_{Y,Y}^+ \cap S_{T,Y}^+) \cup (S_{Y,Y}^- \cap S_{T,Y}^-)\), gives us \(H_{T,Y}^{S,R}\):

\[
H_{T,Y}^{S,R} = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} > 0 \wedge \eta_{T,Y} \neq \frac{\sigma_{TT}}{\sigma_{YT}} \right\}.
\]

**Proof of Proposition 3.** The impact response of variable \(i\) to a non-policy shock is given by:

\[
A_{i,1}^{-1} = \frac{(\sigma_{TT} - \eta_{T,Y}\sigma_{TY})(-\eta_{T,Y}\sigma_{YT}\sigma_{Yi} + \sigma_{Yi}\sigma_{TT} + \eta_{TY}\sigma_{YY}\sigma_{Ti} - \sigma_{YT}\sigma_{Ti})}{(\eta_{YT}^2\sigma_{YY} - 2\eta_{TY}\sigma_{YT} + \sigma_{TT})(-\sigma_{Yi}^2 + \sigma_{YY}\sigma_{TT})}.
\]  

(3.23)

I want to find the set of \(\eta_{T,Y}\) for which:

\[
A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0 \wedge A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0 \wedge A_{i,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0,
\]

or

\[
A_{Y,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0 \wedge A_{T,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0 \wedge A_{i,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0.
\]

I first concentrate on the impact response of variable \(i\). Denote \(S_{i,Y}^+\) as the set of \(\eta_{T,Y}\) for which \(A_{i,Y}^{-1}(\eta_{T,Y}, \Sigma_u) > 0\) and \(S_{i,Y}^-\) as the set of \(\eta_{T,Y}\) for which \(A_{i,Y}^{-1}(\eta_{T,Y}, \Sigma_u) < 0\). The denominator of (3.23) is positive provided that \(|\rho_{YT}| < 1\). In order to determine the sign (3.23) it suffices to analyze its numerator.

**CASE 1**

I study CASE 1 and assume that \(\rho_{Ti} > 0\), \(\rho_{Ti} - \rho_{YT}\rho_{Yi} > 0\), and
\[ \rho_{YT} \rho_{Ti} - \rho_{Yi} < 0 \]. Given these assumptions I find for positive non-policy shocks that:

\[ S_{i,Y}^{+} = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} \in \left( \frac{\sigma_{T} \rho_{YT} \rho_{Ti} - \rho_{Yi}}{\sigma_{Y} \rho_{Ti} - \rho_{YTi}}, \frac{\sigma_{TT}}{\sigma_{Y}} \right) \right\} . \quad (3.24) \]

Note that the first two assumptions guarantee that \( \frac{\sigma_{T} \rho_{YT} \rho_{Ti} - \rho_{Yi}}{\sigma_{Y} \rho_{Ti} - \rho_{YTi}} < \frac{\sigma_{TT}}{\sigma_{Y}} \) and the set \( s_{i,Y}^{+} \) is nonempty. Moreover, the second and third assumptions imply that \( \frac{\sigma_{T} \rho_{YT} \rho_{Ti} - \rho_{Yi}}{\sigma_{Y} \rho_{Ti} - \rho_{YTi}} < 0 \). The intersection of (3.19), (3.20), (3.24) gives the set of \( \eta_{T,Y} \) for which the response of output, tax revenue, and variable \( i \) are positive:

\[ S_{Y,Y}^{+} \cap S_{T,Y}^{+} \cap S_{i,Y}^{+} = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} \in \left( 0, \frac{\sigma_{TT}}{\sigma_{Y}} \right) \right\} . \]

The response of variable \( i \) to a non-policy shock is negative for:

\[ S_{i,Y}^{-} = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} \in \mathbb{R} \setminus \left( \frac{\sigma_{T} \rho_{YT} \rho_{Ti} - \rho_{Yi}}{\sigma_{Y} \rho_{Ti} - \rho_{YTi}}, \frac{\sigma_{TT}}{\sigma_{Y}} \right) \right\} . \quad (3.25) \]

The intersection of (3.21), (3.22), (3.25) gives the set of \( \eta_{T,Y} \) for which for which the response of output, tax revenue, and variable \( i \) are negative:

\[ S_{Y,Y}^{-} \cap S_{T,Y}^{-} \cap S_{i,Y}^{-} = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} > \frac{\sigma_{TT}}{\sigma_{Y}} \right\} . \]

Finally, the union of both intersections, \( (S_{Y,Y}^{+} \cap S_{T,Y}^{+} \cap S_{i,Y}^{+}) \cup (S_{Y,Y}^{-} \cap S_{T,Y}^{-} \cap S_{i,Y}^{-}) \), gives us \( \eta_{T,Y}^{S,R} \):

\[ H_{T,Y}^{S,R} = \left\{ \eta_{T,Y} \in \mathbb{R} : \eta_{T,Y} > 0 \land \eta_{T,Y} \neq \frac{\sigma_{TT}}{\sigma_{Y}} \right\} . \]

**CASE 2**

Next I study CASE 2 and assume that \( \rho_{Ti} < 0 \) and \( \rho_{Ti} - \rho_{TY} \rho_{Yi} < 0 \) and \( \rho_{YT} \rho_{Ti} - \rho_{Yi} < 0 \). Given these assumptions I find for positive non-policy
shocks that:

\[ S_{i,Y}^+ = \left\{ \eta_{T,Y} \in \mathbb{R} : \; \eta_{T,Y} < \frac{\sigma_T \rho_{YT}\rho_{Ti} - \rho_{Yi}}{\sigma_Y \rho_{Ti} - \rho_{YT}\rho_{Yi}} \lor \eta_{T,Y} > \frac{\sigma_{TT}}{\sigma_{TY}} \right\}. \quad (3.26) \]

Note that the first two assumptions guarantee that \( \frac{\sigma_T \rho_{YT}\rho_{Ti} - \rho_{Yi}}{\sigma_Y \rho_{Ti} - \rho_{YT}\rho_{Yi}} < \frac{\sigma_{TT}}{\sigma_{TY}} \). Moreover, the second and third assumptions imply that \( \frac{\sigma_T \rho_{YT}\rho_{Ti} - \rho_{Yi}}{\sigma_Y \rho_{Ti} - \rho_{YT}\rho_{Yi}} > 0 \).

The intersection of (3.19), (3.20), (3.26) gives the set of \( \eta_{T,Y} \) for which the response of output, tax revenue, and variable \( i \) are positive:

\[ S_{Y,Y}^+ \cap S_{T,Y}^+ \cap S_{i,Y}^+ = \left\{ \eta_{T,Y} \in \mathbb{R} : \; \eta_{T,Y} \in \left( 0, \frac{\sigma_T \rho_{YT}\rho_{Ti} - \rho_{Yi}}{\sigma_Y \rho_{Ti} - \rho_{YT}\rho_{Yi}} \right) \right\}. \]

Now I turn to negative non-policy shocks. I find that:

\[ S_{i,Y}^- = \left\{ \eta_{T,Y} \in \mathbb{R} : \; \eta_{T,Y} \in \left( \frac{\sigma_T \rho_{YT}\rho_{Ti} - \rho_{Yi}}{\sigma_Y \rho_{Ti} - \rho_{YT}\rho_{Yi}}, \frac{\sigma_{TT}}{\sigma_{TY}} \right) \right\}. \quad (3.27) \]

The intersection of (3.21), (3.22), (3.27) gives the set of \( \eta_{T,Y} \) for which the response of output, tax revenue, and variable \( i \) are negative:

\[ S_{Y,Y}^- \cap S_{T,Y}^- \cap S_{i,Y}^- = \emptyset. \]

Finally, the sum of both intersections, \((S_{Y,Y}^+ \cap S_{T,Y}^+ \cap S_{i,Y}^+) \cup (S_{Y,Y}^- \cap S_{T,Y}^- \cap S_{i,Y}^-)\), gives us \( H_{T,Y}^{S,R} \):

\[ H_{T,Y}^{S,R} = \left\{ \eta_{T,Y} \in \mathbb{R} : \; \eta_{T,Y} \in \left( 0, \frac{\sigma_T \rho_{YT}\rho_{Ti} - \rho_{Yi}}{\sigma_Y \rho_{Ti} - \rho_{YT}\rho_{Yi}} \right) \right\}. \]

\[ \blacksquare \]

**Proof Proposition 4.** In the spending model the impact response of output \((Y_i)\) to a non-policy shock is given by:

\[ A_{Y,Y}^{-1} (\eta_{G,Y}, \Sigma_u) = \frac{\sigma_{GG} - \eta_{G,Y}\sigma_{YG}}{\sigma_{GG} - 2\eta_{G,Y}\sigma_{YG} + \eta_{G,Y}^2\sigma_{YY}}. \quad (3.28) \]
A. APPENDIX

The impact response of variable  of a non-policy shock is given by:

\[ A_{i,Y}^{-1} = \frac{(\sigma_{GG} - \eta_{G,Y}\sigma_{YG})(-\eta_{G,Y}\sigma_{YG}\sigma_{Yi} + \sigma_{Yi}\sigma_{GG} + \eta_{YG}\sigma_{YY}\sigma_{Gi} - \sigma_{YG}\sigma_{Gi})}{(\eta_{G,Y}\sigma_{YY} - 2\eta_{G,Y}\sigma_{YG} + \sigma_{GG})(-\sigma_{YG}^2 + \sigma_{YY}\sigma_{GG})}. \]

(3.29)

I have to find the set of \( \eta_{G,Y} \) for which:

\[ A_{Y,Y}^{-1}(\eta_{G,Y}, \Sigma_u) > 0 \land A_{i,Y}^{-1}(\eta_{G,Y}, \Sigma_u) > 0, \]

or

\[ A_{Y,Y}^{-1}(\eta_{G,Y}, \Sigma_u) < 0 \land A_{i,Y}^{-1}(\eta_{G,Y}, \Sigma_u) < 0. \]

Denote \( S_{i1}^+ \) as the set of \( \eta_{G,Y} \) for which \( A_{Y,Y}^{-1}(\eta_{G,Y}, \Sigma_u) > 0 \) and \( S_{i1}^+ \) as the set of \( \eta_{G,Y} \) for which \( A_{i,Y}^{-1}(\eta_{G,Y}, \Sigma_u) > 0 \). Denote \( S_{i1}^+ \) as the set of \( \eta_{G,Y} \) for which \( A_{i,Y}^{-1}(\eta_{G,Y}, \Sigma_u) > 0 \) and \( S_{i1}^- \) as the set of \( \eta_{G,Y} \) for which \( A_{i,Y}^{-1}(\eta_{G,Y}, \Sigma_u) < 0 \). Similarly, the denominator of (3.29) is positive provided that \( |\rho_{YG}| < 1 \). So in order to determine the sign (3.28) and (3.29) it suffices to analyze their numerators.

**CASE 1**

I study CASE 1 and assume that \( \rho_{Gi} > 0 \) and \( \rho_{Gi} - \rho_{YG}\rho_{Yi} > 0 \) and \( \rho_{YG}\rho_{Gi} - \rho_{Yi} < 0 \). Given these assumptions I find for positive non-policy shocks that:

\[ S_{i,Y}^+ = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{T,Y} \in \left( \frac{\sigma_{G}\rho_{YG}\rho_{Gi} - \rho_{Yi}}{\sigma_{Y}\rho_{Gi} - \rho_{YG}\rho_{Yi}}, \frac{\sigma_{GG}}{\sigma_{YG}} \right) \right\}, \]

(3.30)

Note that the first two assumptions guarantee that \( \frac{\sigma_{G}\rho_{YG}\rho_{Gi} - \rho_{Yi}}{\sigma_{Y}\rho_{Gi} - \rho_{YG}\rho_{Yi}} < \frac{\sigma_{GG}}{\sigma_{YG}} \) and the set \( S_{i1}^+ \) is nonempty. Moreover, the second and third assumption implies that \( \frac{\sigma_{G}\rho_{YG}\rho_{Gi} - \rho_{Yi}}{\sigma_{Y}\rho_{Gi} - \rho_{YG}\rho_{Yi}} < 0 \). Drawing from the proof of Proposition 2 I find that:

\[ S_{Y,Y}^+ = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} < \frac{\sigma_{GG}}{\sigma_{YG}} \right\}, \]

The intersection of (3.30), (3.30) gives the set of \( \eta_{G,Y} \) for which the output
response and response of variable $i$ are both positive:

$$S_{Y,Y}^+ \cap S_{i,Y}^+ = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} \in \left( \frac{\sigma_G \rho_{YG} \rho_{Gi} - \rho_{Yi}}{\sigma_Y \rho_{Gi} - \rho_{YG} \rho_{Yi}} , \frac{\sigma_{GG}}{\sigma_{YG}} \right) \right\}.$$  

Now I turn to negative non-policy shocks. I find that:

$$S_{i,Y}^- = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{T,Y} \in \mathbb{R} \setminus \left( \frac{\sigma_G \rho_{YG} \rho_{Gi} - \rho_{Yi}}{\sigma_Y \rho_{Gi} - \rho_{YG} \rho_{Yi}} , \frac{\sigma_{GG}}{\sigma_{YG}} \right) \right\}, \quad (3.31)$$

$$S_{Y,Y}^- = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} > \frac{\sigma_{GG}}{\sigma_{YG}} \right\}. \quad (3.32)$$

The intersection of (3.32), (3.31) gives the set of $\eta_{G,Y}$ for which the output response and response of variable $i$ are both negative:

$$S_{Y,Y}^- \cap S_{i,Y}^- = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} > \frac{\sigma_{GG}}{\sigma_{YG}} \right\}.$$  

Finally, the sum of both intersections, $(S_{Y,Y}^+ \cap S_{i,Y}^+) \cup (S_{Y,Y}^- \cap S_{i,Y}^-)$, gives us $H_{G,Y}^{S,R}$:

$$H_{G,Y}^{S,R} = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} > \frac{\sigma_G \rho_{YG} \rho_{Gi} - \rho_{Yi}}{\sigma_Y \rho_{Gi} - \rho_{YG} \rho_{Yi}} \right\}.$$  

**CASE 2**

Next I study CASE 2 and assume that $\rho_{Gi} < 0$ and $\rho_{Gi} - \rho_{YG} \rho_{Yi} < 0$ and $\rho_{YG} \rho_{Gi} - \rho_{Yi} < 0$. Given these assumptions I find for positive non-policy shocks that:

$$S_{i,Y}^+ = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} < \frac{\sigma_G \rho_{YG} \rho_{Gi} - \rho_{Yi}}{\sigma_Y \rho_{Gi} - \rho_{YG} \rho_{Yi}} \lor \eta_{G,Y} > \frac{\sigma_{GG}}{\sigma_{YG}} \right\}. \quad (3.33)$$

Note that the first two assumptions guarantee that $\frac{\sigma_G \rho_{YG} \rho_{Gi} - \rho_{Yi}}{\sigma_Y \rho_{Gi} - \rho_{YG} \rho_{Yi}} < \frac{\sigma_{GG}}{\sigma_{YG}}$. Moreover, the second and third assumptions imply that $\frac{\sigma_G \rho_{YG} \rho_{Gi} - \rho_{Yi}}{\sigma_Y \rho_{Gi} - \rho_{YG} \rho_{Yi}} > 0$. Drawing from the proof of Proposition 2 I find that:
The intersection of (3.34), (3.33) gives the set of \( \eta_{G,Y} \) for which the output response and response of variable \( i \) are both positive:

\[
S_{Y,Y}^+ \cap S_{i,Y}^+ = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} < \frac{\sigma_G \rho_Y \rho_{G,i} - \rho_{Y,i}}{\sigma_Y \rho_{G,i} - \rho_{Y,G} \rho_{Y,i}} \right\}.
\]

Now I turn to negative non-policy shocks. I find that:

\[
S_{i,Y}^- = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} \in \left( \frac{\sigma_G \rho_Y \rho_{G,i} - \rho_{Y,i}}{\sigma_Y \rho_{G,i} - \rho_{Y,G} \rho_{Y,i}} \sigma_{G,Y} \right) \right\},
\]

(3.35)

\[
S_{Y,Y}^- = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} > \frac{\sigma_{G,Y}}{\sigma_Y} \right\}.
\]

(3.36)

The intersection of (3.36), (3.35) gives the set of \( \eta_{G,Y} \) for which the output response and response of variable \( i \) are both negative:

\[
S_{Y,Y}^- \cap S_{i,Y}^- = i \omega.
\]

Finally, the sum of both intersections, \( (S_{Y,Y}^+ \cap S_{i,Y}^+) \cup (S_{Y,Y}^- \cap S_{i,Y}^-) \), gives us \( H_{G,Y}^{S,R} \):

\[
H_{G,Y}^{S,R} = \left\{ \eta_{G,Y} \in \mathbb{R} : \eta_{G,Y} < \frac{\sigma_G \rho_Y \rho_{G,i} - \rho_{Y,i}}{\sigma_Y \rho_{G,i} - \rho_{Y,G} \rho_{Y,i}} \right\}.
\]

\[\blacksquare\]

### A.3 Data and Estimation

I estimate the VAR model using Bayesian techniques. In particular I impose prior distributions on the reduced-form coefficients \( B(L) \) and \( \Sigma_u \) following the methodology discussed in Sims and Zha (1998), with the exception that
I impose priors on a reduced-form rather than on a structural coefficients. I implement this prior, which is a variant of the well-known Minnesota prior, through dummy observations\(^{41}\). The hyper-parameters of the prior are chosen to maximize the posterior marginal data density of the VAR model which, as argued by Del Negro and Schorfheide (2010), tends to work well for inference as well as for forecasting purposes.

All the components of national income are in real per capita terms and are transformed from the nominal values by dividing them by the GDP deflator (NIPA Table 1.1.4, Line 1) and the population measure (NIPA Table 2.1, Line 38). The remaining series are downloaded from Federal Reserve Bank of St. Louis FRED database. The table and row numbers refer to the organization of the data by the BEA. Data are at quarterly frequency from 1947 : 1 to 2010 : 1. I use the logarithm of all variables except the interest rate where I have used the level.

- **GDP**: Gross Domestic Product (NIPA Table 1.1.5, Line 1).
- **Government Spending**: Government consumption (NIPA Table 3.1, Line 16) expenditures and gross investment (NIPA Table 3.1, Line 35).
- **Tax Revenue**: Government current receipts (NIPA Table 3.1, Line 1) minus Current transfer payments (NIPA Table 3.1, Line 17) minus Government interest payments (NIPA Table 3.1, Line 22).
- **Private Consumption**: Personal consumption expenditures (NIPA Table 1.1.5, Line 2).
- **Residential Investment**: Private fixed investment - residential (NIPA Table 1.1.5, Line 12).
- **Non-Residential Investment**: Private fixed investment - non-residential (NIPA Table 1.1.5, Line 9).

\(^{41}\) For a detailed discussion see Del Negro and Schorfheide (2010)
A. APPENDIX

- **CPI** (Fred Series ID: CPIAUCSL): Consumer Price Index For All Urban Consumers (All Items).


- **Stock Market Index**: S&P500 Composite w/GFD extension.

- **Interest Rate** (Fred Series ID: TB3MS): 3-Month Treasury Bill: Secondary Market Rate.

- **Real Wage** (Fred Series ID: COMPRNFB): Nonfarm Business Sector: Real Compensation Per Hour.

The estimates of the output elasticities of tax bases reported in section 3.5 come from the following regressions:

\[
\triangle \log (TB_{i,t}) = \alpha_{i,0} + \eta_{TB,y} \triangle \log (y_t) + \epsilon_t, \tag{3.37}
\]

estimated using Bayesian techniques. Residuals \(\epsilon_t\) are assumed to be autocorrelated. Tax bases are defined as follows:

- Personal Income Tax and Social Security Contributions: Compensation of employees (NIPA Table 1.12, Line 2).

- Corporate Income Tax: Proprietors' income with IVA and CCAdj (NIPA Table 1.12, Line 9) plus Corporate profits with IVA and CCAdj (NIPA Table 1.12, Line 13) plus Rental income of persons with CCAdj (Table 1.12, Line 12).

- Indirect Taxes: Personal consumption expenditures (NIPA Table 1.1.5, Line 2).

- Transfers to Persons: Civilian Unemployment Rate (Fred Series ID: Unrate_Q).
Regressions (3.37) have been estimated using as explanatory variables either real output or the output gap (as done by the OECD), where potential output is 'Real Potential Gross Domestic Product’ (GDPPOT). Left-hand side variables are either measured in nominal terms (as done by the OECD), or in real terms dividing the original series by the GDP deflator. These variations have only a minor impact on the coefficients.

The series for the elasticity of the US Federal Personal Income Tax estimated using the TAXSIM model is available at http://www.nber.org/~taxsim/elas/. The series for OECD elasticities personal income tax and social security contributions are extracted from the original papers cited in the main text.
A. APPENDIX

A.4 Tables and Figures

<table>
<thead>
<tr>
<th></th>
<th>Impact Multiplier</th>
<th>Elasticity</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>min</td>
<td>max</td>
<td>$\eta_{P,Y}^{\min}$</td>
<td>$\eta_{P,Y}^{\max}$</td>
<td></td>
</tr>
<tr>
<td>2-equation VAR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tax</td>
<td>$-0.69$</td>
<td>$0.69$</td>
<td>$-3.07$</td>
<td>$1.50$</td>
<td>$6.07$</td>
</tr>
<tr>
<td>Spending</td>
<td>$-1.54$</td>
<td>$1.54$</td>
<td>$1.90$</td>
<td>$0.29$</td>
<td>$-1.31$</td>
</tr>
<tr>
<td>11-equation VAR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tax</td>
<td>$-0.69$</td>
<td>$0.69$</td>
<td>$-2.94$</td>
<td>$1.61$</td>
<td>$6.18$</td>
</tr>
<tr>
<td>Spending</td>
<td>$-1.52$</td>
<td>$1.52$</td>
<td>$2.02$</td>
<td>$0.40$</td>
<td>$-1.22$</td>
</tr>
</tbody>
</table>

Table 1: Results in this table are based on the estimation of three reduced-form model: A two-equation tax model consisting of output and tax revenue (row 1); A two-equation spending model consisting of output and government spending (row 2); An 11-equation VAR model described in section 2.3 in the main text (rows 3,4). Columns 2 and 3 report the minimum and maximum tax and spending multiplier which can be obtained varying the output elasticity of tax revenue and government spending. Columns 4, 5, and 6 report the output elasticity of fiscal variables associated to the smallest multiplier, to the zero multiplier, and to the largest multiplier. Multipliers represent the effect in dollars of a one-dollar tax cut and a one dollar spending increase shocks.
### Table 2: Output Elasticity of Tax Revenue

<table>
<thead>
<tr>
<th>Method</th>
<th>Output Elasticity of Tax Revenue</th>
<th>$\Pi_{0}^{T,T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Median 68% C.S.</td>
<td>Median 68% C.S.</td>
</tr>
<tr>
<td>Recursive(T,Y)</td>
<td>1.62 (1.32; 1.91)</td>
<td>0.00 (0.11; 0.29)</td>
</tr>
<tr>
<td>Traditional SVAR</td>
<td>2.26 (0.52; 0.88)</td>
<td>0.00 (0.11; 0.29)</td>
</tr>
<tr>
<td>Sign Restrictions</td>
<td>4.84 (-0.04; 0.66)</td>
<td>0.40 (-0.04; 0.66)</td>
</tr>
<tr>
<td>Penalty Function</td>
<td>3.32 (0.41; 0.52)</td>
<td>0.46 (0.41; 0.52)</td>
</tr>
</tbody>
</table>

Table 2: This table compares impact tax multipliers computed using four identification schemes. For each identification scheme I compute the implied restriction on the output elasticity of tax revenue. If an identification scheme restricts the elasticity to a set, I report the median elasticity and the 68% credible set. The multiplier represents the effect in dollars of a one-dollar tax cut shock. I report the median multiplier and the 68% credible set.

### Table 3: Output Elasticity of Spending

<table>
<thead>
<tr>
<th>Method</th>
<th>Output Elasticity of Spending</th>
<th>$\Pi_{0}^{T,G}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Median 68% C.S.</td>
<td>Median 68% C.S.</td>
</tr>
<tr>
<td>Recursive(G,Y)</td>
<td>0.00 -</td>
<td>0.70 (0.52; 0.88)</td>
</tr>
<tr>
<td>Traditional SVAR</td>
<td>0.00 -</td>
<td>0.70 (0.52; 0.88)</td>
</tr>
<tr>
<td>Sign Restrictions</td>
<td>0.38 (-1.35; 1.35)</td>
<td>0.02 (-1.35; 1.35)</td>
</tr>
<tr>
<td>Penalty Function</td>
<td>0.20 (0.23; 0.49)</td>
<td>0.36 (0.23; 0.49)</td>
</tr>
</tbody>
</table>

Table 3: This table compares impact spending multipliers computed using four identification schemes. For each identification scheme I compute the implied restriction on the output elasticity of government spending. If an identification scheme restricts the elasticity to a set, I report the median elasticity and the 68% credible set. The multiplier represents the effect in dollars of a one-dollar spending (increase) shock. I report the median multiplier and the 68% credible set.
A. APPENDIX

\[
\begin{array}{|c|c|}
\hline
\text{Method} & P_r\left(\Pi_{0}^{Y,G} > \Pi_{0}^{Y,T}\right) \\
\hline
\text{Recursive} & 1.00 \\
\text{Traditional SVAR} & 0.99 \\
\text{Sign Restrictions} & 0.45 \\
\text{Penalty Function} & 0.22 \\
\hline
\end{array}
\]

Table 4: This table compares the probability of the impact spending multiplier to be larger than the impact tax multiplier, computed using four identification schemes. The tax multiplier represents the effect in dollars of a one-dollar tax cut shock. The spending multiplier represents the effect in dollars of a one dollar spending increase shock.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Method} & \text{Output Elasticity of Tax Revenue} & \Pi_{0}^{Y,T} & \text{Median} & 68\% \text{ C.S.} \\
\hline
\text{Prior OECD} & 2.26 & (2.16; 2.37) & 0.19 & (0.09; 0.28) \\
\text{Prior OECD+NBER} & 1.81 & (1.73; 1.91) & 0.06 & (-0.04; 0.15) \\
\text{Prior DSGE} & 1.62 & (1.45; 1.92) & 0.01 & (-0.10; 0.12) \\
\hline
\end{array}
\]

Table 5: This table compares impact spending multipliers computed using three prior distributions of the output elasticity of tax revenue. The derivation of the prior distributions is described in section 3.5. I report the median elasticity and the 68\% credible set. The multiplier represents the effect in dollars of a one-dollar tax cut shock. I report the median multiplier and the 68\% credible set.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Method} & \text{Output Elasticity of Spending} & \Pi_{0}^{Y,G} & \text{Median} & 68\% \text{ C.S.} \\
\hline
\text{Prior data} & 0.09 & (0.03; 0.13) & 0.54 & (0.33; 0.75) \\
\text{Prior DSGE} & -0.06 & (-0.10; -0.03) & 0.79 & (0.60; 0.98) \\
\hline
\end{array}
\]

Table 6: This table compares impact spending multipliers computed using two prior distributions of the output elasticity of government spending. The derivation of the prior distributions is described in section 3.5. I report the median elasticity and the 68\% credible set. The multiplier represents the effect in dollars of a one-dollar spending (increase) shock. I report the median multiplier and the 68\% credible set.
Table 7: This table reports the probability of the impact spending multiplier to be larger than the tax multiplier, computed using alternative prior distributions on the output elasticities of tax revenue and government spending. The derivation of the prior distributions is described in section 3.5. The tax multiplier represents the effect in dollars of a one-dollar tax cut shock. The spending multiplier represents the effect in dollars of a one-dollar spending (increase) shock.

<table>
<thead>
<tr>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prior OECD</td>
</tr>
<tr>
<td>Prior OECD + NBER</td>
</tr>
<tr>
<td>Prior DSGE</td>
</tr>
</tbody>
</table>

Table 8: Column 2 reports the tax revenue elasticity to tax base ($\eta_{T,TB}$) for five tax categories. These elasticities are constructed from micro data and legislation by the OECD. Columns 4 and 5 report the output elasticity of tax base to output ($\eta_{TB,Y}$). These elasticities are estimated using Bayesian linear regressions (3.37). I report the median elasticity and the 68% credible set.

<table>
<thead>
<tr>
<th>$\eta_{T,TB}$</th>
<th>$\eta_{TB,Y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tax Elasticity</td>
<td>Tax Base Median</td>
</tr>
<tr>
<td>Personal Income</td>
<td>2.54</td>
</tr>
<tr>
<td>Social Security</td>
<td>0.95</td>
</tr>
<tr>
<td>Corporate</td>
<td>1</td>
</tr>
<tr>
<td>Indirect</td>
<td>1</td>
</tr>
<tr>
<td>Transfers to Persons</td>
<td>1</td>
</tr>
</tbody>
</table>
### A. APPENDIX

#### Table 9: Calibration of deep parameters for the DSGE model from Leeper et al. (2009a)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior Distribution</th>
<th>Density</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_k$</td>
<td>Gamma</td>
<td>1</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td>$\varphi_l$</td>
<td>Gamma</td>
<td>0.5</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>$\varphi_g$</td>
<td>Gamma</td>
<td>0.07</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>Calibration</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td></td>
<td></td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>$\tau^k$</td>
<td></td>
<td></td>
<td>0.185</td>
<td></td>
</tr>
<tr>
<td>$\tau^f$</td>
<td></td>
<td></td>
<td>0.223</td>
<td></td>
</tr>
</tbody>
</table>

#### Table 10: This table reports the response of output components to fiscal shocks. Shocks are identified computing prior distributions from data as illustrated in section 5, prior Data 2. Multipliers represent the effect in dollars of a one-dollar tax cut and spending (increase) shocks. I report the median multiplier and the 68% credible set.

<table>
<thead>
<tr>
<th></th>
<th>Spending Increase</th>
<th>Tax Cut</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>68% C.S.</td>
<td>Median</td>
</tr>
<tr>
<td>Consumption</td>
<td>-0.09</td>
<td>(-0.20; 0.03)</td>
</tr>
<tr>
<td>Non-Res. Investment</td>
<td>-0.05</td>
<td>(-0.10; 0.00)</td>
</tr>
<tr>
<td>Real Wage</td>
<td>-0.04</td>
<td>(-0.07; -0.01)</td>
</tr>
</tbody>
</table>
Figure 1: Top Panel: Impact tax multiplier as a function of the output elasticity of tax revenue. Reduced-form coefficients are OLS estimates from the 2-equation VAR tax model. The multiplier represents the effect on output in dollars of a one-dollar tax cut shock. Bottom Panel: Impact spending multiplier as a function of the output elasticity of government spending. Reduced-form coefficients are OLS estimates from the 2-equation VAR spending model. The multiplier represents the effect on output in dollars of a one-dollar spending (increase) shock.
Figure 2: Top Panel: Impact tax multiplier as a function of the output elasticity of tax revenue. Reduced-form coefficients are OLS estimates from the 2-equation VAR tax model (blue solid line), and an 11-equation VAR model (black dashed line). The multiplier represents the effect on output in dollars of a one-dollar tax cut shock. Bottom Panel: Impact spending multiplier as a function of the output elasticity of government spending. Reduced-form coefficients are OLS estimates from the 2-equation VAR tax model (blue solid line), and an 11-equation VAR model (black dashed line). The multiplier represents the effect on output in dollars of a one-dollar spending (increase) shock.
Figure 3: Top Panel: Impact tax multiplier as a function of the output elasticity of tax revenue. Reduced-form coefficients are OLS estimates from the 11-equation VAR model. The multiplier represents the effect on output in dollars of a one-dollar tax cut shock. Bottom Panel: Impact spending multiplier as a function of the output elasticity of government spending. Reduced-form coefficients are OLS estimates from an 11-equation VAR model. The multiplier represents the effect on output in dollars of a one-dollar spending (increase) shock.
Figure 4: Elasticity of personal income tax to earnings from the OECD (blue dashed-line), and elasticity of personal income tax and social security contributions estimated using the TAXSIM model (red solid line).
Figure 5: Tax multipliers four-quarter (top panel), eight-quarter (central panel), and twelve-quarter (bottom panel) after the shock. Reduced-form coefficients are OLS estimates from the 11-equation VAR model. The multiplier represents the effect on output in dollars of a one-dollar tax cut shock.
Figure 6: Spending multipliers four-quarter (top panel), eight-quarter (central panel), and twelve-quarter (bottom panel) after the shock. Reduced-form coefficients are OLS estimates from the 11-equation VAR model. The multiplier represents the effect on output in dollars of a one-dollar tax cut shock.
Figure 7: Multipliers represent the effect in dollars on output of one-dollar spending increase or tax cut shocks. Multipliers are computed according to the following formula: \( \text{Output response} = \frac{\text{Initial fiscal shock}}{(\text{Average fiscal variable share of output})} \). The reduced-form model is an 11-equation VAR estimated using Bayesian techniques. The blue solid line is the median multiplier, the red dashed lines are 16th and 84th percentiles.
Figure 8: Probability of the spending multiplier to be larger than the tax multiplier. The reduced-form model is an 11-equation VAR estimated using Bayesian techniques.
Figure 9: Impact responses of private consumption, non-residential investment, and real wages to a spending shock (left column) and to a tax shock (right column). Responses of consumption and investment are expressed as multipliers (dollar response to a spending increase or a tax cut of size one dollar). The response of real wages is expressed in changes.
Chapter 4

Computing DSGE Models with Recursive Preferences and Stochastic Volatility*

4.1 Introduction

This paper compares different solution methods for computing the equilibrium of dynamic stochastic general equilibrium (DSGE) models with recursive preferences and stochastic volatility (SV). Both features have become very popular in finance and macroeconomics as modeling devices to account for business cycle fluctuations and asset pricing. Recursive preferences, as those first proposed by Kreps and Porteus (1978) and later generalized by Epstein and Zin (1989, 1991) and Weil (1990), are attractive for two reasons. First, they allow us to separate risk aversion and intertemporal elasticity of

*This chapter is co-authored with Jesús Fernández-Villaverde, Juan F. Rubio-Ramírez and Wen Yao. We thank Michel Juillard for his help with computational issues and Larry Christiano, Dirk Krueger, Pawel Zabczyk, and participants at the several seminars for comments. Beyond the usual disclaimer, we must note that any views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Atlanta or the Federal Reserve System. Finally, we also thank the NSF for financial support.
COMPUTING DSGE MODELS

substitution (EIS). Second, they offer the intuitive appeal of having preferences for an early or later resolution of uncertainty (see the reviews by Backus et al. (2004, 2007), and Hansen et al. (2007), for further details and references). SV generates aggregate fluctuations with time-varying volatility, a basic property of many time series such as output (see the review by Fernández-Villaverde and Rubio-Ramírez, 2010) and adds extra flexibility in accounting for asset pricing patterns. In fact, in an influential paper, Bansal and Yaron (2004) have argued that the combination of recursive preferences and SV is the key for their proposed mechanism, long-run risk, to be successful in explaining asset pricing.

But despite the popularity of these issues, little is known about the numerical properties of the different solution methods that solve equilibrium models with recursive preferences and SV. For example, we do not know how well value function iteration (VFI) performs or how good local approximations are as compared to global ones. Similarly, if we want to estimate the model, we need to assess what solution method is sufficiently reliable yet quick enough to make the exercise feasible. More important, the most common solution algorithm in the DSGE literature, (log-) linearization, cannot be applied, since it makes us miss the whole point of recursive preferences or SV: the resulting (log-) linear decision rules are certainty equivalent and do not depend on risk aversion or volatility. This paper attempts to fill this gap in the literature, and therefore, it complements previous work by Aruoba et al. (2006), where a similar exercise is performed with the neoclassical growth model with a CRRA utility function and constant volatility.

We solve and simulate the model using four main approaches: perturbation (of second- and third-order), Chebyshev polynomials, and VFI. Thus, we span most of the relevant methods in the literature. Our results provide a strong guess of how some other methods not covered here, such as finite elements, would work (rather similar to Chebyshev polynomials but more computationally intensive). We report results for a benchmark calibration
of the model and for alternative calibrations that change the variance of the productivity shock, the risk aversion, and the intertemporal elasticity of substitution. In that way, we study the performance of the methods both for cases close to the CRRA utility function with constant volatility and for highly non-linear cases far away from the CRRA benchmark. For each method, we compute decision rules, the value function, the ergodic distribution of the economy, business cycle statistics, the welfare costs of aggregate fluctuations, and asset prices. Moreover, we evaluate the accuracy of the solution by reporting Euler equation errors.

We highlight four main results from our exercise. First, all methods provide a high degree of accuracy. Thus, researchers who stay within our set of solution algorithms can be confident that their quantitative answers are sound.

Second, perturbations deliver a surprisingly high level of accuracy with considerable speed. Both second- and third-order perturbations performs remarkably well in terms of accuracy for the benchmark calibration, being competitive with VFI or Chebyshev polynomials. For this calibration, a second-order perturbation that runs in a fraction of a second does nearly as well in terms of the average Euler equation error as a VFI that takes ten hours to run. Even in the extreme calibration with high risk aversion and high volatility of productivity shocks, perturbation works at a more than acceptable level. Since, in practice, perturbation methods are the only computationally feasible method for solving the medium-scale DSGE models used for policy analysis that have dozens of state variables (as in Smets and Wouters, 2007), this finding has an outmost applicability. Moreover, since implementing second- and third-order perturbations is feasible with off-the-shelf software like Dynare, which requires minimum programming knowledge by the user, our findings may induce many researchers to explore recursive preferences and/or SV in further detail. Two final advantages of perturbation are that, often, the perturbed solution provides insights about the economics of
the problem and that it might be an excellent initial guess for VFI or for Chebyshev polynomials.

Third, Chebyshev polynomials provide a terrific level of accuracy with a reasonable computational burden. When accuracy is most required and the dimensionality of the state space is not too high, they are the obvious choice. Unfortunately, Chebyshev polynomials suffer from the curse of dimensionality and, for more involved models, we would need to apply some aggressive interpolation scheme as in Krueger and Kubler (2006).

Fourth, we were disappointed by the poor performance of VFI which, compared with Chebyshev, could not achieve a high accuracy even with a large grid. This suggests that we should relegate VFI to solving those problems where non-differentiability complicate the application of the previous methods.

The rest of the paper is organized as follows. In section 4.2, we present our test model. Section 4.3 describes the different solution methods used to approximate the decision rules of the model. Section 4.4 discusses the calibration of the model. Section 4.5 reports numerical results and section 4.6 concludes the paper. An appendix provides some additional details.

4.2 The Model

We use the stochastic neoclassical growth model with recursive preferences and SV in the process for technology as our test case. We select this model for three reasons. First, it is the workhorse of modern macroeconomics. Even more complicated New Keynesian models with real and nominal rigidities, such as those in Woodford (2003) or Christiano et al. (2005), are built around the core of the neoclassical growth model. Thus, any lesson learned with it is likely to have a wide applicability. Second, the model is, except for the form of the utility function and the process for SV, the same test case as in Aruoba et al. (2006). This provides us with a set of results to compare to
4.2. THE MODEL

our findings. Three, the introduction of recursive preferences and SV make the model both more non-linear (and hence, a challenge for different solution algorithms) and potentially more relevant for practical use. For example, and as mentioned in the introduction, Bansal and Yaron (2004) have emphasized the importance of the combination of recursive preferences and time-varying volatility to account for asset prices.

The description of the model is straightforward, and we just go through the details required to fix notation. There is a representative household that has preferences over streams of consumption, $c_t$, and leisure, $1 - l_t$, that are representable by a recursive function of the form:

$$U_t = \max_{c_t, l_t} \left[ (1 - \beta) \left( c_t^\varphi (1 - l_t)^{1 - \varphi} \right)^{\frac{1 - \gamma}{\varphi}} + \beta \left( \mathbb{E}_t U_{t+1}^{1 - \gamma} \right)^{\frac{1}{1 - \gamma}} \right]$$  \hspace{1cm} (4.1)

The parameters in these preferences include $\beta$, the discount factor, $\varphi$, which controls labor supply, $\gamma$, which controls risk aversion, and:

$$\theta = \frac{1 - \gamma}{1 - \varphi}$$

where $\psi$ is the EIS. The parameter $\theta$ is an index of the deviation with respect to the benchmark CRRA utility function (when $\theta = 1$, we are back in that CRRA case where the inverse of the EIS and risk aversion coincide).

The household’s budget constraint is given by:

$$c_t + i_t + \frac{b_{t+1}}{R_t} = w_t l_t + r_t k_t + b_t$$

where $i_t$ is investment, $R_t$ is the risk-free gross interest rate, $b_t$ is the holding of an uncontingent bond that pays 1 unit of consumption good at time $t + 1$, $w_t$ is the wage, $l_t$ is labor, $r_t$ is the rental rate of capital, and $k_t$ is capital. Asset markets are complete and we could have also included in the budget constraint the whole set of Arrow securities. Since we have a representative
household, this is not necessary because the net supply of any security is zero. Households accumulate capital according to the law of motion $k_{t+1} = (1 - \delta)k_t + i_t$ where $\delta$ is the depreciation rate.

The final good in the economy is produced by a competitive firm with a Cobb-Douglas technology $y_t = e^{z_t k_t^{1-\xi}}$ where $z_t$ is the productivity level that follows:

$$z_t = \lambda z_{t-1} + e^{\sigma_t} \varepsilon_t, \varepsilon_t \sim N(0, 1).$$

The innovation $\varepsilon_t$ is scaled by a SV level $\sigma_t$, which evolves as:

$$\sigma_t = (1 - \rho)\overline{\sigma} + \rho \sigma_{t-1} + \eta \omega_t, \omega_t \sim N(0, 1)$$

where $\overline{\sigma}$ is the unconditional mean level of $\sigma_t$, $\rho$ is the persistence of the processes, and $\eta$ is the standard deviation of the innovations to $\sigma_t$. Our specification is parsimonious and it introduces only two new parameters, $\rho$ and $\eta$. At the same time, it captures some important features of the data (see a detailed discussion in Fernández-Villaverde and Rubio-Ramírez, 2010). Another important point is that, with SV, we have two innovations, an innovation to technology, $\varepsilon_t$, and an innovation to the standard deviation of technology, $\omega_t$. Finally, the economy must satisfy the aggregate resource constraint $y_t = c_t + i_t$.

The definition of equilibrium is standard and we skip it in the interest of space. Also, both welfare theorems hold, a fact that we will exploit by jumping back and forth between the solution of the social planner’s problem and the competitive equilibrium. However, this is only to simplify our derivations. We can easily apply the solution methods described below to solve problems that are not Pareto optimal. In those cases, the household will still have a

---

1Stationarity is the natural choice for our exercise. If we had a deterministic trend, we would only need to adjust $\beta$ in our calibration below (and the results would be nearly identical). If we had a stochastic trend, we would need to rescale the variables by the productivity level and solve the transformed problem. However, in this case, it is well known that the economy fluctuates less than when $\lambda < 1$, and therefore, all solution methods would be closer, limiting our ability to appreciate differences in their performance.
value function that will allow us to derive the appropriate optimality conditions that we can complete with other equilibrium conditions such as market clearing.

Thus, an alternative way to write this economy is to look at the value function representation of the social planner’s problem in terms of its three state variables, capital $k_t$, productivity $z_t$, and volatility, $\sigma_t$:

$$V(k_t, z_t, \sigma_t) = \max_{c_t, l_t} \left[ (1 - \beta) \left( c_t^\upsilon(1 - l_t)^{1-\upsilon} \right)^{\frac{1-\gamma}{\sigma}} + \beta \left( \mathbb{E}_t V^{1-\gamma}(k_{t+1}, z_{t+1}, \sigma_{t+1}) \right)^{\frac{1}{1-\gamma}} \right]$$

s.t. $c_t + k_{t+1} = e^{z_t} k_t^{1-\xi} + (1 - \delta) k_t$

$z_t = \lambda z_{t-1} + \sigma_t \varepsilon_t, \varepsilon_t \sim \mathcal{N}(0, 1)$

$\sigma_t = (1 - \rho) \bar{\sigma} + \rho \sigma_{t-1} + \eta \omega_t, \omega_t \sim \mathcal{N}(0, 1)$.

Then, we can find the pricing kernel of the economy

$$m_{t+1} = \frac{\partial V_t / \partial c_{t+1}}{\partial V_t / \partial c_t}.$$  

Now, note that:

$$\frac{\partial V_t}{\partial c_t} = (1 - \beta) V_t^{1-\frac{1-\gamma}{\sigma}} \upsilon \left( c_t^\upsilon(1 - l_t)^{1-\upsilon} \right)^{\frac{1-\gamma}{\sigma}}$$

and:

$$\frac{\partial V_t}{\partial c_{t+1}} = \beta V_{t+1}^{1-\frac{1-\gamma}{\sigma}} \left( \mathbb{E}_t V_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \mathbb{E}_t \left( (1 - \beta) V_{t+1}^{1-\frac{1-\gamma}{\sigma}} \upsilon \left( c_{t+1}^\upsilon(1 - l_{t+1})^{1-\upsilon} \right)^{\frac{1-\gamma}{\sigma}} \right)$$

where in the last step we use the result regarding $\partial V_t / \partial c_t$ forwarded by one period. Cancelling redundant terms, we get:

$$m_{t+1} = \frac{\partial V_t / \partial c_{t+1}}{\partial V_t / \partial c_t} = \beta \left( \frac{c_t^\upsilon(1 - l_t)^{1-\upsilon}}{c_{t+1}^\upsilon(1 - l_{t+1})^{1-\upsilon}} \right)^{\frac{1-\gamma}{\sigma}} \frac{c_t}{c_{t+1}} \left( \frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{\frac{1-\gamma}{\sigma}}. \quad (4.2)$$
This equation shows how the pricing kernel is affected by the presence of recursive preferences. If $\theta = 1$, the last term,

$$\left( \frac{V_{t+1}^{1-\gamma}}{E_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\sigma}}$$  \hspace{1cm} (4.3)$$

is equal to 1 and we get back the pricing kernel of the standard CRRA case. If $\theta \neq 1$, the pricing kernel is twisted by (4.3).

We identify the net return on equity with the marginal net return on investment:

$$R^k_{t+1} = \zeta e^{z_{t+1}} k_{t+1}^{\xi - 1} l_{t+1}^{\Lambda - \zeta} - \delta$$

with expected return $E_t[R^k_{t+1}]$.

### 4.3 Solution Methods

We are interested in comparing different solution methods to approximate the dynamics of the previous model. Since the literature on computational methods is large, it would be cumbersome to review every proposed method. Instead, we select those methods that we find most promising.

The first method we pick is perturbation (introduced by Judd and Guu (1992, 1997), and particularly well explained in Schmitt-Grohé and Uribe, 2004). Perturbation algorithms build a Taylor series expansion of the agents' decision rules around an appropriate point (usually the steady state of the economy) and a perturbation parameter. In many situations, perturbation methods are very fast and, despite their local nature, to be highly accurate in a large range of values of the state variables (Aruoba et al., 2006). This means that, in practice, perturbations are the only method that can handle models with dozens of state variables within any reasonable amount of time. Moreover, perturbation often provides insights into the structure of the solution and on the economics of the model. Finally, linearization and
4.3. SOLUTION METHODS

log-linearization, the most common solution methods for DSGE models, are a particular case of a perturbation of first order.

We implement a second- and a third-order perturbation of our model. A first-order perturbation is useless for our investigation: the resulting decision rules are certainty equivalent and, therefore, they depend on the EIS but not on $\gamma$ or on $\sigma_t$. In other words, the first-order decision rules of the model with recursive preferences coincide with the decision rules of the model with CRRA preferences with the same EIS and constant volatility for any value of $\gamma$ or $\sigma_t$. We need to go, at least, to second-order decision rules to have terms that depend on $\gamma$ or on $\sigma_t$ and, hence, allow recursive preferences or SV to play a role. Since the accuracy of second-order decision rules may not be high enough and, in addition, we want to explore time-varying risk premia, we also compute a third-order perturbation. As we will document below, a third-order perturbation provides enough accuracy without unnecessary complications. Thus, we do not need to go to higher orders.

The second method is a projection algorithm with Chebyshev polynomials (Judd, 1992). Projection algorithms build approximated decision rules that minimize a residual function that measures the distance between the left- and right-hand side of the equilibrium conditions of the model. Projection methods are attractive because they offer a global solution over the whole range of the state space. Their main drawback is that they suffer from an acute curse of dimensionality that makes it quite challenging to extend them to models with many state variables. Among the many different types of projection methods, (Aruoba et al., 2006) show that Chebyshev polynomials are particularly efficient. Other projection methods, such as finite elements or parameterized expectations, tend to perform somewhat worse than Chebyshev polynomials, and therefore, in the interest of space, we do not consider them.

Finally, we compute the model using VFI (Epstein and Zin, 1989, show that a version of the contraction mapping theorem holds in the value function
of the problem with recursive preferences). VFI is slow and it suffers as well from the curse of dimensionality, but it is safe, reliable, and we know its numerical convergence properties. Thus, it is a natural default algorithm for the solution of DSGE models.

We describe now each of the different methods in more detail. Then, we calibrate the model and present numerical results.

### 4.3.1 Perturbation

We start by explaining how to use a perturbation approach to solve DSGE models using the value function of the household. We are not the first to explore the perturbation of value function problems. Judd (1998) already presents the idea of perturbing the value function instead of the equilibrium conditions of a model. Unfortunately, he does not elaborate much on the topic. Schmitt-Grohé and Uribe (2006) employ a perturbation approach to find a second-order approximation to the value function that allows them to rank different fiscal and monetary policies in terms of welfare. However, we follow Binsbergen and Rubio-Ramírez (2009) in their emphasis on the generality of the approach.

To illustrate the procedure, we limit our exposition to deriving the second-order approximation to the value function and the decision rules of the agents. Higher-order terms are derived analogously, but the algebra becomes too

---

2The perturbation method is linked with Benigno and Woodford (2006) and Hansen and Sargent (1995). Benigno and Woodford present a linear-quadratic (LQ) approximation to solve optimal policy problems when the constraints of the problem are non-linear (see also Levine and Pierse, 2007). In particular, Benigno and Woodford find the correct local welfare ranking of different policies. Our perturbation can also deal with non-linear constraints and obtains the correct local approximation to welfare and policies, but with the advantage that it is easily generalizable to higher-order approximations. Hansen and Sargent (1995) modify the LQ problem to adjust for risk. In that way, they can handle some versions of recursive utilities. Hansen and Sargent’s method, however, requires imposing a tight functional form for future utility and to surrender the assumption that risk-adjusted utility is separable across states of the world. Perturbation does not suffer from those limitations.
4.3. SOLUTION METHODS

cumbersome to be developed explicitly (in our application, the symbolic algebra is undertaken by Mathematica, which automatically generates Fortran 95 code that we can evaluate numerically). Hopefully, our steps will be enough to allow the reader to understand the main thrust of the procedure and obtain higher-order approximations by herself.

First, we rewrite the exogenous processes in terms of a perturbation parameter $\chi$:

$$z_t = \lambda z_{t-1} + \chi e^{\alpha_t} \varepsilon_t$$
$$\sigma_t = (1 - \rho) \tilde{\sigma} + \rho \sigma_{t-1} + \chi \eta \omega_t.$$  

When $\chi = 1$, which is just a normalization, we are dealing with the stochastic version of the model. When $\chi = 0$, we are dealing with the deterministic case with steady state $k_{ss}$, $z_{ss} = 0$, and $\sigma_{ss} = \tilde{\sigma}$. Then, we can write the social planner’s value function, $V(k_t, z_t, \sigma_t; \chi)$, and the decision rules for consumption, $c(k_t, z_t, \sigma_t; \chi)$, investment, $i(k_t, z_t, \sigma_t; \chi)$, capital, $k(k_t, z_t, \sigma_t; \chi)$, and labor, $l(k_t, z_t, \sigma_t; \chi)$, as a function of the states and the perturbation parameter.

Second, we note that, under differentiability assumptions, the second-order Taylor approximation of the value function around the steady state $(k_{ss}, 0, \sigma_{ss}; 0)$ is:

$$V(k_t, z_t, \sigma_t; \chi) \simeq V_{ss} + V_{1,ss} (k_t - k_{ss}) + V_{2,ss} z_t + V_{3,ss} (\sigma_t - \sigma_{ss}) + V_{4,ss} \chi$$

$$+ \frac{1}{2} V_{11,ss} (k_t - k_{ss})^2 + \frac{1}{2} V_{12,ss} (k_t - k_{ss}) z_t$$

$$+ \frac{1}{2} V_{13,ss} (k_t - k_{ss}) (\sigma_t - \sigma_{ss}) + \frac{1}{2} V_{14,ss} (k_t - k_{ss}) \chi$$

$$+ \frac{1}{2} V_{21,ss} z_t (k_t - k_{ss}) + \frac{1}{2} V_{22,ss} z_t^2 + \frac{1}{2} V_{23,ss} z_t (\sigma_t - \sigma_{ss}) + \frac{1}{2} V_{24,ss} z_t \chi$$

$$+ \frac{1}{2} V_{31,ss} (\sigma_t - \sigma_{ss}) (k_t - k_{ss}) + \frac{1}{2} V_{32,ss} (\sigma_t - \sigma_{ss}) z_t$$

$$+ \frac{1}{2} V_{33,ss} (\sigma_t - \sigma_{ss})^2 + \frac{1}{2} V_{34,ss} (\sigma_t - \sigma_{ss}) \chi.$$
where we define $V_{ss} = V (k_{ss}, 0, \sigma_{ss}; 0)$, $V_{i,ss} = V_i (k_{ss}, 0, \sigma_{ss}; 0)$ for $i = \{1, 2, 3, 4\}$, and $V_{ij,ss} = V_{ij} (k_{ss}, 0, \sigma_{ss}; 0)$ for $i, j = \{1, 2, 3, 4\}$. We can extend this notation to higher-order derivatives of the value function. This expansion could also be performed around a different point of the state space, such as the mode of the ergodic distribution of the state variables. In section 4.5, we discuss this point further.

By certainty equivalence, we have that $V_{4,ss} = V_{14,ss} = V_{41,ss} = V_{24,ss} = V_{42,ss} = V_{34,ss} = V_{43,ss} = 0$. In fact, all the terms in odd powers of $\chi$ have coefficient values equal to zero. Moreover, it is also the case that $V_{3,ss} = V_{13,ss} = V_{31,ss} = V_{33,ss} = 0$. Finally, taking advantage of the equality of cross-derivatives, and setting $\chi = 1$, the approximation we look for has the simpler form:

$$V (k_t, z_t, \sigma_t; 1) \simeq V_{ss} + V_{1,ss} (k_t - k_{ss}) + V_{2,ss} z_t$$

$$+ \frac{1}{2} V_{11,ss} (k_t - k_{ss})^2 + \frac{1}{2} V_{22,ss} z_t^2 + V_{12,ss} (k_t - k_{ss}) z_t$$

$$+ V_{23,ss} z_t (\sigma_t - \sigma_{ss}) + \frac{1}{2} V_{44,ss}$$

Binsbergen et al. (2009) demonstrate that $\gamma$ does not affect the values of any of the coefficients except $V_{44,ss}$ and also that $V_{44,ss} \neq 0$. Hence, our approximation is different from the one resulting from the standard linear-quadratic one, where all the constants disappear. This result is intuitive, since the value function of a risk-adverse agent is in general affected by uncertainty and we want to have an approximation with terms that capture this effect and allow for the appropriate welfare ranking of decision rules. Indeed, $V_{44,ss}$ has a straightforward interpretation. At the deterministic steady state, we have:

$$V (k_{ss}, 0, \sigma_{ss}; 1) \simeq V_{ss} + \frac{1}{2} V_{44,ss}$$
4.3. SOLUTION METHODS

Hence \( \frac{1}{2} V_{44,ss} \) is a measure of the welfare cost of the business cycle, that is, of how much utility changes when the variance of the productivity shocks is at steady state value \( \sigma_{ss} \) instead of zero (note that this quantity is not necessarily negative). This term is an accurate evaluation of the third-order of the welfare cost of business cycle fluctuations because all of the third-order terms in the approximation of the value function either have zero coefficient values (for example, \( V_{44,ss} = 0 \)) or drop when evaluated at the deterministic steady state.

This cost of the business cycle can easily be transformed into consumption equivalent units. We can compute the percentage decrease in consumption \( \tau \) that will make the household indifferent between consuming \( (1 - \tau) c_{ss} \) units per period with certainty or \( c_t \) units with uncertainty. To do so, note that the steady-state value function is just \( V_{ss} = c_{ss}^{\upsilon} (1 - l_{ss})^{1-\upsilon} \), which implies that:

\[
c_{ss}^{\upsilon} (1 - l_{ss})^{1-\upsilon} + \frac{1}{2} V_{44,ss} = ((1 - \tau) c_{ss})^{\upsilon} (1 - l_{ss})^{1-\upsilon}
\]

or:

\[
V_{ss} + \frac{1}{2} V_{44,ss} = (1 - \tau)^{\upsilon} V_{ss}
\]

Then:

\[
\tau = 1 - \left[ 1 + \frac{1}{2} \frac{V_{44,ss}}{V_{ss}} \right]^{\frac{1}{\upsilon}}.
\]

We are perturbing the value function in levels of the variables. However, there is nothing special about levels and we could have done the same in logs (a common practice when linearizing DSGE models) or in any other function of the states. These changes of variables may improve the performance of perturbation (Fernández-Villaverde and Rubio-Ramírez, 2006). By doing the perturbation in levels, we are picking the most conservative case for perturbation. Since one of the conclusions that we will reach from our numerical results is that perturbation works surprisingly well in terms of accuracy, that conclusion will only be reinforced by an appropriate change of variables.\(^3\)

\(^3\)This comment begets the question, nevertheless, of why we did not perform a per-
The decision rules can be expanded in the same way. For example, the second-order approximation of the decision rule for consumption is, under differentiability assumptions:

\[
\begin{align*}
& c(k_t, z_t, \sigma_t; \chi) \simeq c_{ss} + c_{1,ss}(k_t - k_{ss}) + c_{2,ss}z_t + c_{3,ss}(\sigma_t - \sigma_{ss}) + c_{4,ss}\chi \\
& + \frac{1}{2}c_{11,ss}(k_t - k_{ss})^2 + \frac{1}{2}c_{12,ss}(k_t - k_{ss})z_t + \frac{1}{2}c_{13,ss}(k_t - k_{ss})(\sigma_t - \sigma_{ss}) \\
& + \frac{1}{2}c_{14,ss}(k_t - k_{ss})\chi \\
& + \frac{1}{2}c_{21,ss}z_t(k_t - k_{ss}) + \frac{1}{2}c_{22,ss}z_t^2 + \frac{1}{2}c_{23,ss}z_t(\sigma_t - \sigma_{ss}) + \frac{1}{2}c_{24,ss}z_t\chi \\
& + \frac{1}{2}c_{31,ss}(\sigma_t - \sigma_{ss})(k_t - k_{ss}) + \frac{1}{2}c_{32,ss}(\sigma_t - \sigma_{ss})z_t \\
& + \frac{1}{2}c_{33,ss}(\sigma_t - \sigma_{ss})^2 + \frac{1}{2}c_{34,ss}(\sigma_t - \sigma_{ss})\chi \\
& + \frac{1}{2}c_{41,ss}\chi(k_t - k_{ss}) + \frac{1}{2}c_{42,ss}\chi z_t + \frac{1}{2}c_{43,ss}\chi(\sigma_t - \sigma_{ss}) + \frac{1}{2}c_{44,ss}\chi^2
\end{align*}
\]

where \(c_{ss} = c(k_{ss}, 0; 0), c_{i,ss} = c_i(k_{ss}, 0; 0)\) for \(i = \{1, 2, 3\}\), \(c_{ij,ss} = c_{ij}(k_{ss}, 0; 0)\) for \(i, j = \{1, 2, 3\}\). Again, applying the result that any term on odd powers of \(\chi\) is zero, certainty equivalence, and the fact that \(c_{3,ss} = c_{13,ss} = c_{31,ss} = c_{33,ss} = 0\), we get the much simpler expression:

\[
\begin{align*}
& c(k_t, z_t, \sigma_t; 1) \simeq c_{ss} + c_{1,ss}(k_t - k_{ss}) + c_{2,ss}z_t \\
& + \frac{1}{2}c_{11,ss}(k_t - k_{ss})^2 + \frac{1}{2}c_{22,ss}z_t^2 + c_{12,ss}(k_t - k_{ss})z_t \\
& + \frac{1}{2}c_{23,ss}z_t(\sigma_t - \sigma_{ss}) + \frac{1}{2}c_{44,ss}
\end{align*}
\]

As with the approximation of the value function, (Binsbergen and Rubio-Ramírez, 2009) show that \(\gamma\) does not affect the values of any of the coefficients.

---

\*turbation in logs, since many economists will find it more natural than in levels. Our experience with the CRRA utility case (Aruoba et al., 2006) and some computations with recursive preferences not included in the paper suggest that a perturbation in logs does slightly worse than a perturbation in levels.
except $c_{4,ss}$. This term is a constant that captures precautionary behavior caused by risk. This observation tells us two facts. First, a linear approximation to the decision rule does not depend on $\gamma$ (it is certainty equivalent), and therefore, if we are interested in recursive preferences, we need to go at least to a second-order approximation. Second, given some fixed parameter values, the difference between the second-order approximation to the decision rules of a model with CRRA preferences and a model with recursive preferences is a constant. This constant generates a second, indirect effect, because it changes the ergodic distribution of the state variables and, hence, the points where we evaluate the decision rules along the equilibrium path. These arguments demonstrate how perturbation methods can provide analytic insights beyond computational advantages and help in understanding the numerical results in Tallarini (2000). In the third-order approximation, all of the terms on functions of $\chi^2$ depend on $\gamma$.

Following the same steps, we can derive the decision rules for labor, investment, and capital. In addition we have functions that give us the evolution of other variables of interest, such as the pricing kernel or the risk-free gross interest rate $R_f^t$. All of these functions have the same structure and properties regarding $\gamma$ as the decision rule for consumption. In the case of functions pricing assets, the second-order approximation generates a constant risk premium, while the third-order approximation creates a time-varying risk premium.

Once we have reached this point, there are two paths we can follow to solve for the coefficients of the perturbation. The first procedure is to write down the equilibrium conditions of the model plus the definition of the value function. Then, we take successive derivatives in this augmented set of equilibrium conditions and solve for the unknown coefficients. This approach, which we call equilibrium conditions perturbation (ECP), gets us, after $n$ iterations, the $n$-th-order approximation to the value function and to the decision rules.
A second procedure is to take derivatives of the value function with respect to states and controls and use those derivatives to find the unknown coefficient. This approach, which we call value function perturbation (VFP), delivers after \((n + 1)\) steps, the \((n + 1)\)-th order approximation to the value function and the \(n\)-th order approximation to the decision rules.\(^4\) Loosely speaking, ECP undertakes the first step of VFP by hand by forcing the researcher to derive the equilibrium conditions.

The ECP approach is simpler but it relies on our ability to find equilibrium conditions that do not depend on derivatives of the value function. Otherwise, we need to modify the equilibrium conditions to include the definitions of the derivatives of the value function. Even if this is possible to do (and not particularly difficult), it amounts to solving a problem that is equivalent to VFP. This observation is important because it is easy to postulate models that have equilibrium conditions where we cannot get rid of all the derivatives of the value function (for example, in problems of optimal policy design). ECP is also faster from a computational perspective. However, VFP is only trivially more involved because finding the \((n + 1)\)-th-order approximation to the value function on top of the \(n\)-th order approximation requires nearly no additional effort.

The algorithm presented here is based on the system of equilibrium equations derived using the ECP. In the appendix, we derive a system of equations using the VFP. We take the first-order conditions of the social planner. First, with respect to consumption today:

\[
\frac{\partial V_t}{\partial c_t} - \lambda_t = 0
\]

where \(\lambda_t\) is the Lagrangian multiplier associated with the resource constraint.

\(^4\)The classical strategy of finding a quadratic approximation of the utility function to derive a linear decision rule is a second-order example of VFP (Anderson et al., 1996). A standard linearization of the equilibrium conditions of a DSGE model when we add the value function to those equilibrium conditions is a simple case of ECP. This is done, for instance, in Schmitt-Grohé and Uribe (2006).
4.3. SOLUTION METHODS

Second, with respect to capital:

\[-\lambda_t + E_t \lambda_{t+1} \left( \zeta e^{z_{t+1}} k_{t+1}^{\zeta - 1} l_{t+1}^{1-\zeta} + 1 - \delta \right) = 0.\]

Third, with respect to labor:

\[\frac{1 - \upsilon}{\upsilon} \frac{c_t}{(1-l_t)} = (1 - \zeta) e^{z_t} k_t^{\zeta} l_t^{1-\zeta}.\]

Then, we have \( E_t m_{t+1} \left( \zeta e^{z_{t+1}} k_{t+1}^{\zeta - 1} l_{t+1}^{1-\zeta} + 1 - \delta \right) = 1 \) where \( m_{t+1} \) was derived above in equation (4.2). Note that, as we explained above, the derivatives of the value function in (4.2) are eliminated.

Once we substitute for the pricing kernel, the augmented equilibrium conditions are:

\[
\begin{align*}
V_t - \left[ (1 - \beta) \left( c_t^\upsilon (1 - l_t)^{1-\upsilon} \right)^{\frac{1-\gamma}{\sigma}} + \beta \left( E_t V_t^{1-\gamma} (k_{t+1}, z_{t+1}) \right) \right]^{\frac{1}{1-\gamma}} &= 0 \\
E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\frac{1-\gamma}{\sigma}} \left( \frac{V_{t+1}^{1-\gamma}}{E_t V_t^{1-\gamma}} \right) \right]^{-\frac{1}{1-\gamma}} \left( \zeta e^{z_{t+1}} k_{t+1}^{\zeta - 1} l_{t+1}^{1-\zeta} + 1 - \delta \right) - 1 &= 0 \\
\frac{1 - \upsilon}{\upsilon} \frac{c_t}{(1-l_t)} &= (1 - \zeta) e^{z_t} k_t^{\zeta} l_t^{1-\zeta} = 0 \\
E_t/\beta \left( \frac{c_{t+1}}{c_t} \right)^{\frac{1-\gamma}{\sigma}} \left( \frac{V_{t+1}^{1-\gamma}}{E_t V_t^{1-\gamma}} \right) \left( R_t^I - 1 \right) &= 0 \\
c_t + i_t - e^{z_t} k_t^{\zeta} l_t^{1-\zeta} &= 0 \\
k_{t+1} - i_t - (1 - \delta) k_t &= 0
\end{align*}
\]

plus the law of motion for productivity and volatility and where we have dropped the max operator in front of the value function because we are already evaluating it at the optimum. In more compact notation,

\[F(k_t, z_t, \sigma_t; \chi) = 0\]
where $F$ is a 8-dimensional function (and where all the endogenous variables in the previous equation are not represented explicitly because they are functions themselves of the states) and $0$ is the vectorial zero.

In steady state, $m_{ss} = \beta$ and the set of equilibrium conditions simplifies to:

$$V_{ss} = c_{ss} (1 - l_{ss})^{1-v}$$
$$\left(\zeta l_{ss}^{1-v} + 1 - \delta\right) = \frac{1}{\beta}$$
$$\frac{1 - v}{v} \frac{c_{ss}}{(1-l_{ss})} = (1 - \zeta)k_{ss}l_{ss}^{1-\zeta}$$
$$R_{ss}^l = \frac{1}{\beta}$$
$$c_{ss} + i_{ss} = k_{ss}^{1-\zeta}$$
$$i_{ss} = \delta k_{ss}$$

a system of 6 equations on 6 unknowns, $V_{ss}, c_{ss}, k_{ss}, i_{ss}, l_{ss}$, and $R_{ss}^l$ that can be easily solved (see the appendix for the derivations). This steady state is identical to the steady state of the real business cycle model with a standard CRRA utility function and no SV.

To find the first-order approximation to the value function and the decision rules, we take first derivatives of the function $F$ with respect to the states $(k_t, z_t, \sigma_t)$ and to the perturbation parameter $\chi$ and evaluate them at the deterministic steady state $(k_{ss}, 0, \sigma_{ss}; 0)$ that we just found:

$$F_i (k_{ss}, 0, \sigma_{ss}; 0) = 0 \text{ for } i = \{1, 2, 3, 4\}.$$  

This step gives us 32 different first derivatives (8 equilibrium conditions times the 4 variables of $F$). Since each dimension of $F$ is equal to zero for all possible values of $k_t, z_t, \sigma_t, \chi$, their derivatives must also be equal to zero. Therefore, once we substitute in the values of the steady state and forget about the exogenous processes (which we do not need to solve for), we have a quadratic system of 24 equations on 24 unknowns: $V_{i,ss}, c_{i,ss}, i_{i,ss},$
4.3. SOLUTION METHODS

$k_{i,ss}$, $l_{i,ss}$, and $R_{i,ss}^f$ for $i = \{1, 2, 3, 4\}$. One of the solutions is an unstable root of the system that violates the transversality condition of the problem and we eliminate it. Thus, we keep the solution that implies stability. In the solution, it is easy to see that $V_{4,ss} = c_{4,ss} = k_{4,ss} = i_{4,ss} = l_{4,ss} = R_{4,ss}^f = 0$, that is, we have certainty equivalence. This result is not a surprise and it could have been guessed from a reading of Schmitt-Grohe and Uribe (2004).

To find the second-order approximation, we take derivatives on the first derivatives of the function $F$, again with respect to the states and the perturbation parameter:

$$F_{ij}(k_{ss}, 0, \sigma_{ss}; 0) = 0$$

for $i, j = \{1, 2, 3, 4\}$.

This step gives us a new system of equations. Then, we plug in the terms that we already know from the steady state and from the first-order approximation and we get that the only unknowns left are the second-order terms of the value function and other functions of interest. Quite conveniently, this system of equations is linear and it can be solved quickly. Repeating these steps (taking higher-order derivatives, plugging in the terms already known, and solving for the remaining unknowns), we can get any arbitrary order approximation. For simplicity, and since we checked that we were already obtaining a high accuracy, we decided to stop at a third-order approximation (we are particularly interested in applying the perturbation for estimation purposes and we want to document how a third-order approximation is accurate enough for many problems without spending too much time deriving higher-order terms).

4.3.2 Projection

Projection methods take basis functions to build an approximated value function and decision rules that minimize a residual function defined by the augmented equilibrium conditions of the model. There are two popular methods
for choosing basis functions: finite elements and the spectral method. We will present only the spectral method. There are several reasons for this: first, in the neoclassical growth model the decision rules and value function are smooth and spectral methods provide an excellent approximation (Aruoba et al., 2006). Second, spectral methods allow us to use a large number of basis functions, with the consequent high accuracy. Third, spectral methods are easier to implement. Their main drawback is that since they approximate the solution globally, if the decision rules display a rapidly changing local behavior or kinks, it may be difficult for this scheme to capture those local properties.

Our target is to solve the decision rule for labor and the value function \( \{l_t, V_t\} \) from the two conditions:

\[
\mathcal{H}(l_t, V_t) = \left[ u_{c,t} - \beta \left( \mathbb{E}_t V_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}-1} \mathbb{E}_t \left[ V_{t+1}^{\frac{(1-\gamma)(\theta-1)}{\theta}} u_{c,t+1} \left( R_k^{k+1} \right) \right] V_t - \left[ (1 - \beta)(c_t^\gamma (1 - l_t^{1-\gamma}))^{\frac{1-\gamma}{\theta}} + \beta \mathbb{E}_t (V_{t+1}^{1-\gamma})^{\frac{1}{\theta}} \right] \right] = 0
\]

where, to save on notation, we define \( V_t = V(k_t, z_t, \sigma_t) \) and:

\[
u_{c,t} = \frac{1 - \gamma}{\theta} \left( c_t^\gamma (1 - l_t^{1-\gamma})^{\frac{1-\gamma}{\theta}} \right)
\]

Then, from the static condition

\[
c_t = \frac{\nu}{1 - \nu}(1 - \zeta) e^{z_t} k_t^\zeta l_t^{-\zeta}(1 - l_t)
\]

and the resource constraint, we can find \( c_t \) and \( k_{t+1} \).

Spectral methods solve this problem by specifying the decision rule for labor and the value function \( \{l_t, V_t\} \) as linear combinations of weighted basis functions:

\[
l(k_t, z_j, \sigma_m; \rho) = \sum_i \rho_{ijm}^l \psi_i(k_t)
\]
4.3. SOLUTION METHODS

\[ V(k_t, z_j, \sigma_m; \rho) = \sum_i \rho_{ijm}^V \psi_i(k_t) \]

where \( \{\psi_i(k)\}_{i=1,\ldots,n_k} \) are the \( n_k \) basis functions that we will use for our approximation along the capital dimension and

\[ \rho = \{\rho_{ijm}^l, \rho_{ijm}^V\}_{i=1,\ldots,n_k; j=1,\ldots,J; m=1,\ldots,M} \]

are unknown coefficients to be determined. In this expression, we have discretized the stochastic processes \( \sigma_t \) for volatility and \( z_t \) for productivity using Tauchen’s (1986) method as follows. First, we have a grid of \( M \) points \( G_\sigma = \{e^{\sigma_1}, e^{\sigma_2}, \ldots, e^{\sigma_M}\} \) for \( \sigma_t \) and a transition matrix \( \Pi^M \) with generic element \( \pi_{i,j}^M = \text{Prob}(e^{\sigma_{t+1}} = e^{\sigma_i} | e^{\sigma_t} = e^{\sigma_j}) \). Then, for each \( M \) point, we find a grid with \( J \) points \( G_m^z = \{z_{m1}^1, z_{m2}^1, \ldots, z_{mJ}^1\} \) for \( z_t \) and transition matrices \( \Pi_{J,m} \) with generic element \( \pi_{i,j}^{J,m} = \text{Prob}(z_{m+1}^j = z_{mi}^m | z_{mi}^m = z_{m}^m) \). Values for the decision rule outside the grids \( G_\sigma \) and \( G_m^z \) can be approximated by interpolation.

We make the grids for \( z_t \) depend on the level of volatility \( m \) to adapt the accuracy of Tauchen’s procedure to each conditional variance (although this forces us to interpolate when we switch variances). Since we set \( J = 25 \) and \( M = 5 \), the approximation is quite accurate along the productivity axis (we explored other choices of \( J \) and \( M \) to be sure that our choice was sensible).

A common choice for the basis functions are Chebyshev polynomials because of their flexibility and convenience. Since their domain is \([-1,1]\), we need to bound capital to the set \([k, \bar{k}]\), where \( k \) (\( \bar{k} \)) is chosen sufficiently low (high) to bind only with extremely low low probability, and define a linear map from those bounds into \([-1,1]\). Then, we set \( \psi_i(k) = \tilde{\psi}_i(\phi_k(k_i)) \) where \( \tilde{\psi}_i(\cdot) \) are Chebyshev polynomials and \( \phi_k(k_i) \) is the linear mapping from \([k, \bar{k}]\) to \([-1,1]\).

By plugging \( l(k_t, z_j, \sigma_m; \rho) \) and \( V(k_t, z_j, \sigma_m; \rho) \) into \( \mathcal{H}(l_t, V_t) \), we find the residual function:

\[ \mathcal{R}(k_t, z_j, \sigma_m; \rho) = \mathcal{H}(l(k_t, z_j, \sigma_m; \rho), V(k_t, z_j, \sigma_m; \rho)) \]
We determine the coefficients $\rho$ to get the residual function as close to 0 as possible. However, to do so, we need to choose a weight of the residual function over the space $(k_t, z_j, \sigma_m)$. A collocation point criterion delivers the best trade-off between speed and accuracy (Fornberg, 1998) by making the residual function exactly equal to zero in \( \{k_i\}_{i=1}^{n_k} \) roots of the \( n_k \)-th order Chebyshev polynomial and in the Tauchen points (also, by the Chebyshev interpolation theorem, if an approximating function is exact at the roots of the \( n_k \)-th order Chebyshev polynomial, then, as \( n_k \to \infty \), the approximation error becomes arbitrarily small). Therefore, we just need to solve the following system of \( n_k \times J \times M \times 2 \) equations:

$$ R(k_i, z_j, \sigma_m; \rho) = 0 \text{ for any } i, j, m \text{ collocation points} $$

on \( n_k \times J \times M \times 2 \) unknowns \( \rho \). We solve this system with a Newton method and an iteration based on the increment of the number of basis functions. First, we solve a system with only three collocation points for capital and 125 (125 = 25 \times 5) points for technology. Then, we use that solution as a guess for a system with more collocation points for capital (with the new coefficients being guessed to be equal to zero) and iterate on the procedure. We stop the iteration when we have 11 polynomials in the capital dimension (therefore, in the last step we solve for 2,750 = 11 \times 25 \times 5 \times 2 coefficients). The iteration is needed because otherwise the residual function is too cumbersome to allow for direct solution of the 2,750 final coefficients.

### 4.3.3 Value Function Iteration

Our final solution method is VFI. Since the dynamic algorithm is well known, our presentation is most brief. Consider the following Bellman operator:

$$ TV(k_t, z_t, \sigma_t) = \max_{c_t, l_t} \left[ (1 - \beta) \left( c_t^\nu (1 - l_t)^{1-\nu} \right)^{\frac{1-\gamma}{1-\nu}} + \beta \mathbb{E}_t V^{1-\gamma}(k_{t+1}, z_{t+1}, \sigma_{t+1}) \right]^{\frac{\beta}{1-\gamma}} $$

s.t. \( c_t + k_{t+1} = e^{zt} k_t^{\xi} t_t^{1-\zeta} + (1 - \delta) k_t \)
To solve for this Bellman operator, we define a grid on capital, $G_k = \{k_1, k_2, \ldots, k_M\}$, a grid on volatility and on the productivity level. The grid on capital is just a uniform distribution of points over the capital dimension. As we did for projection, we set a grid $G_\sigma = \{e^{\sigma_1}, e^{\sigma_2}, \ldots, e^{\sigma_M}\}$ for $\sigma_t$ and a transition matrix $\Pi^M$ for volatility and $M$ grids $G_z^m = \{z_1^m, z_2^m, \ldots, z_J^m\}$ for $z_t$ and transition matrixes $\Pi^{J,m}$ using Tauchen’s (1986) procedure. The algorithm to iterate on the value function for this grid is:

1. Set $n = 0$ and $V^0(k_t, z_t, \sigma_t) = c^v_{ss} (1 - l_{ss})^{1-v}$ for all $k_t \in G_k$ and all $z_t \in G_z$.
2. For every $\{k_t, z_t, \sigma_t\}$, use Newton method to find $c_t^*, l_t^*, k_{t+1}^*$ that solve:

$$c_t = \frac{v}{1 - v} (1 - \zeta) e^{z_t} k_t^\xi l_t^{1-\zeta} (1 - l_t)$$

$$(1 - \beta)^{\frac{1-v}{\sigma}} c_t (1 - l_t)^{1-v} = \beta \mathbb{E}_t \left( V^{n+1}_{t+1} \right)^{1-\gamma} \mathbb{E}_t \left[ \left( V_{t+1}^n \right)^{-\gamma} V_{t+1}^{n+1} \right]$$

$$c_t + k_{t+1} = e^{z_t} k_t^\xi l_t^{1-\zeta} + (1 - \delta)k_t$$

3. Construct $V^{n+1}$ from the Bellman equation:

$$V^{n+1} = ((1 - \beta)(c_t^v (1 - l_t^*)^{1-v})^{\frac{1-\gamma}{\sigma}} + \beta \mathbb{E}_t(V(k_{t+1}, z_{t+1}, \sigma_{t+1})^{1-\gamma}))^{\frac{1}{1-\gamma}}$$

4. If $\frac{|V^{n+1} - V^n|}{|V^n|} \geq 1.0 e^{-7}$, then $n = n + 1$ and go to 2. Otherwise, stop.

To accelerate convergence and give VFI a fair chance, we implement a multi-grid scheme as described by Chow and Tsitsiklis (1991). We start by iterating on a small grid. Then, after convergence, we add more points to the grid and

$$z_t = \lambda z_{t-1} + e^{\sigma} \varepsilon_t, \varepsilon_t \sim \mathcal{N}(0, 1)$$

$$\sigma_t = (1 - \rho)\sigma + \rho \sigma_{t-1} + \eta \omega_t, \omega_t \sim \mathcal{N}(0, 1).$$
recompute the Bellman operator using the previously found value function as an initial guess (with linear interpolation to fill the unknown values in the new grid points). Since the previous value function is an excellent grid, we quickly converge in the new grid. Repeating these steps several times, we move from an initial 23,000-point grid into a final one with 375,000 points (3,000 points for capital, 25 for productivity, and 5 for volatility).

4.4 Calibration

We now select a benchmark calibration for our numerical computations. We follow the literature as closely as possible. We set the discount factor $\beta = 0.991$ to generate an annual interest rate of around 3.6 percent. We set the parameter that governs labor supply, $\theta = 0.357$, to get the representative household to work one-third of its time. The elasticity of output to capital, $\zeta = 0.3$, matches the labor share of national income. A value of the depreciation rate $\delta = 0.0196$ matches the ratio of investment-output. Finally, $\lambda = 0.95$ and $\log \sigma = 0.007$ are standard values for the stochastic properties of the Solow residual of the U.S. economy. For the SV process, we pick $\rho = 0.9$ and $\eta = 0.06$, which generate changes in volatility that resemble the observations of the last decades, which display persistent increases and falls of volatility of around 50 percent.

Since we want to explore the dynamics of the model for a range of values that encompasses all the estimates from the literature, we select four values for the parameter that controls risk aversion, $\gamma$, 2, 5, 10, and 40, and two values for EIS $\psi$, 0.5, and 1.5, which bracket most of the values used in the literature (although many authors prefer smaller values for $\psi$, we found that the simulation results for smaller $\psi$’s do not change much from the case when $\psi = 0.5$). We then compute the model for all eight combinations of values of $\gamma$ and $\psi$, that is $\{2, 0.5\}$, $\{5, 0.5\}$, $\{10, 0.5\}$, and so on. When $\psi = 0.5$ and $\gamma = 2$, we are back in the standard CRRA case. However, in the interest of
space, we will report only a limited subset of results that we find are the most interesting ones.

We pick as the benchmark case the calibration \( \{ \gamma, \psi, \log \sigma, \eta \} = \{ 5, 0.5, 0.007, 0.06 \} \). These values reflect an EIS centered around the median of the estimates in the literature, a reasonably high level of risk aversion, and the observed volatility of productivity shocks. To check robustness, we increase, in the extreme case, the risk aversion, the average standard deviation of the productivity shock, and the standard deviation of the innovations to volatility to \( \{ \gamma, \psi, \log \sigma, \eta \} = \{ 40, 0.5, 0.021, 0.1 \} \). This case combines levels of risk aversion that are in the upper bound of all estimates in the literature with huge productivity shocks. Therefore, it pushes all solution methods to their limits, in particular, making life hard for perturbation since the interaction of large precautionary behavior induced by \( \gamma \) and large shocks will move the economy far away from the deterministic steady state. We leave the discussion of the effects of \( \psi = 1.5 \) for the robustness analysis at the end of the next section.

4.5 Numerical Results

In this section we report our numerical findings. First, we present and discuss the computed decision rules. Second, we show the results of simulating the model. Third, we report the Euler equation errors. Fourth, we discuss the effects of changing the EIS and the perturbation point. Finally, we discuss implementation and computing time.

4.5.1 Decision Rules

One of our first results is the decision rules and the value function of the agent. Figure 1 plots the decision rules for consumption, labor supply, capital, and the value function in the benchmark case when \( z_t = 0 \) and \( \sigma_t = \bar{\sigma} \) over a capital interval centered on the steady-state level of capital of 9.54 with a
width of \( \pm 40\% \), \([5.72,13.36]\). We selected an interval for capital big enough to encompass all the simulations in our sample. Similar figures could be plotted for other values of \( z_t \) and \( \sigma_t \). We omit them because of space considerations.

Since all methods provide nearly indistinguishable answers, we observe only one line in all figures. It is possible to appreciate very tiny differences in labor supply between second-order perturbation and the other methods only when capital is far from its steady-state level. Monotonicity of the decision rules is preserved by all methods. We must be cautious, however, mapping differences in choices into differences in utility. The Euler error function below provides a better view of the welfare consequences of different approximations.

We see bigger differences in the decision rules and value functions as we increase the risk aversion and the variance of innovations. Figure 2 plots the decision rules and value functions for the extreme calibration. In this figure, we change the interval reported because, owing to the high variance of the calibration, the equilibrium paths fluctuate through much wider ranges of capital.

We highlight several results. First, all the methods deliver similar results in our original interval for the benchmark calibration. Second, as we go far away from the steady state, VFI and the Chebyshev polynomial still overlap with each other (and, as shown by our Euler error computations below, we can roughly take them as the “exact” solution), but second- and third-order approximations start to deviate. Third, the decision rule for consumption and the value function approximated by the third-order perturbation changes from concavity into convexity for values of capital bigger than 15. This phenomenon (also documented in Aruoba et al., 2006) is due to the poor performance of local approximation when we move too far away from the expansion point and the polynomials begin to behave wildly. In any case, this issue is irrelevant because, as we will show below, the problematic region is visited with nearly zero probability.
4.5. NUMERICAL RESULTS

4.5.2 Simulations

Applied economists often characterize the behavior of the model through statistics from simulated paths of the economy. We simulate the model, starting from the deterministic steady state, for 10,000 periods, using the decision rules for each of the eight combinations of risk aversion and EIS discussed above. To make the comparison meaningful, the shocks are common across all paths. We discard the first 1,000 periods as a burn-in to eliminate the transition from the deterministic steady state of the model to the middle regions of the ergodic distribution of capital. This is usually achieved in many fewer periods than the ones in our burn-in, but we want to be conservative in our results. The remaining observations constitute a sample from the ergodic distribution of the economy.

For the benchmark calibration, the simulations from all of the solution methods generate almost identical equilibrium paths (and therefore we do not report them). We focus instead on the densities of the endogenous variables as shown in figure 3. Given the low risk aversion and SV of the productivity shocks, all densities are roughly centered around the deterministic steady state value of the variable. For example, the mean of the distribution of capital is only 0.2 percent higher than the deterministic value. Also, capital is nearly always between 8.5 and 10.5. This range will be important below to judge the accuracy of our approximations.

Table 2 reports business cycle statistics and, because DSGE models with recursive preferences and SV are often used for asset pricing, the average and variance of the (quarterly) risk-free rate and return on capital. Again, we see that nearly all values are the same, a simple consequence of the similarity of the decision rules.

The welfare cost of the business cycle is reported in Table 3 in consumption equivalent terms. The computed costs are actually negative. Besides the Jensen’s effect on average productivity, this is also due to the fact that when we have leisure in the utility function, the indirect utility function may be
convex in input prices (agents change their behavior over time by a large amount to take advantage of changing productivity). Cho and Cooley (2000) present a similar example. Welfare costs are comparable across methods. Remember that the welfare cost of the business cycle for the second- and third-order perturbations is the same because the third-order terms all drop or are zero when evaluated at the steady state.

When we move to the extreme calibration, we see more differences. Figure 4 plots the histograms of the simulated series for each solution method. Looking at quantities, the histograms of consumption, output, and labor are the same across all of the methods. The ergodic distribution of capital puts nearly all the mass between values of 6 and 15. This considerable move to the right in comparison with figure 3 is due to the effect of precautionary behavior in the presence of high risk aversion, large productivity shocks, and high SV. Capital also goes down more than in the benchmark calibration because of large, persistent productivity shocks, but not nearly as much as it increases when shocks are positive.

Table 4 reports business cycle statistics. Differences across methods are minor in terms of means (note that the mean of the risk-free rate is lower than in the benchmark calibration because the extra accumulation of capital induced by precautionary behavior). In terms of variances, the second-order perturbation produces less volatility than all other methods. This suggests that a second-order perturbation may not be good enough if we face high variance of the shocks and/or high risk aversion. A third-order perturbation, in comparison, eliminates most of the differences and delivers nearly the same implications as Chebyshev polynomials or VFI.

Finally, Table 5 presents the welfare cost of the business cycle. Now, in comparison with the benchmark calibration, the welfare cost of the business cycle is positive and significant, slightly above 1.1 percent. This is not a surprise, since we have both a large risk aversion and productivity shocks with an average standard deviation three times as big as the observed one.
All methods deliver numbers that are close.

### 4.5.3 Euler Equation Errors

While the plots of the decision rules and the computation of densities and business cycle statistics that we presented in the previous subsection are highly informative, it is also important to evaluate the accuracy of each of the procedures. Euler equation errors, introduced by Judd (1992), have become a common tool for determining the quality of the solution method. The idea is to observe that, in our model, the intertemporal condition:

$$u_{c,t} = \beta (\mathbb{E}_t V_{t+1}^{1-\gamma})^{\frac{1}{\theta}} \mathbb{E}_t \left( V_{t+1}^{(\gamma-1)(1-\theta)} \right) u_{c,t+1} R(k_t, z_t, \sigma_t; z_{t+1}, \sigma_{t+1})$$  \hspace{1cm} (4.4)

where $R(k_t, z_t, \sigma_t; z_{t+1}, \sigma_{t+1}) = 1 + \zeta e^{\gamma+1}k_t^{\gamma-1}l_{t+1}^{1-\gamma} - \delta$ is the gross return of capital given states $k_t$, $z_t$, $\sigma_t$, and realizations $z_{t+1}$ and $\sigma_{t+1}$ should hold exactly for any given $k_t$, and $z_t$. However, since the solution methods we use are only approximations, there will be an error in (4.4) when we plug in the computed decision rules. This Euler equation error function $EE^i(k_t, z_t, \sigma_t)$ is defined, in consumption terms:

$$EE^i(k_t, z_t, \sigma_t) = 1 - \frac{\beta (\mathbb{E}_t (V_{t+1}^{1-\gamma})^{\frac{1}{\theta}} \mathbb{E}_t \left( V_{t+1}^{(\gamma-1)(1-\theta)} \right) u_{c,t+1} R(k_t, z_t, \sigma_t; z_{t+1}, \sigma_{t+1})}{c_t} \left( \frac{1}{1 - \theta} \right)^{\frac{1}{\theta}}$$

This function determines the (unit free) error in the Euler equation as a fraction of the consumption given the current states and solution method $i$. Following Judd and Guu (1997), we can interpret this error as the optimization error incurred by the use of the approximated decision rule and we report the absolute errors in base 10 logarithms to ease interpretation. Thus, a value of -3 means a $1 mistake for each $1000 spent, a value of -4 a $1
mistake for each $10,000 spent, and so on.

Figure 5 displays a transversal cut of the errors for the benchmark calibration when $z = 0$ and $\sigma_t = \bar{\sigma}$. Other transversal cuts at different technology and volatility levels reveal similar patterns. The first lesson from figure 5 is that all methods deliver high accuracy. We know from figure 3 that capital is nearly always between 8.5 and 10.5. In that range, the (log10) Euler equation errors are at most -5, and most of the time they are even smaller. For instance, the second- and third-order perturbations have an Euler equation error of around -7 in the neighborhood of the deterministic steady state, VFI of around -6.5, and Chebyshev an impressive -11/-13. The second lesson from figure 5 is that, as expected, global methods (Chebyshev and VFI) perform very well in the whole range of capital values, while perturbations deteriorate as we move away from the steady state. For second-order perturbation, the Euler error in the steady state is almost four orders of magnitudes smaller than on the boundaries. Third-order perturbation is around half an order of magnitude more accurate than second-order perturbation over the whole range of values (except in a small region close to the deterministic steady state).

There are two complementary ways to summarize the information from Euler equation error functions. First, in the second column of table 6, we report the maximum error in our interval (capital between 75 percent and 125 percent of the steady state and the grids for productivity and volatility). The maximum Euler error is useful because it bounds the mistake owing to the approximation. Both perturbations have a maximum Euler error of around -2.7, VFI of -3.1, and Chebyshev, an impressive -9.8. We read this column as indicating that all methods perform adequately. The second procedure for summarizing Euler equation errors is to integrate the function with respect to the ergodic distribution of capital and productivity to find the average error.\textsuperscript{5} We can think of this exercise as a generalization of the Den Haan–

\textsuperscript{5}There is the technical consideration of which ergodic distribution to use for this task,
Marcet test (Den Haan and Marcet, 1994). This integral is a welfare measure of the loss induced by the use of the approximating method. We report our results in the third column of table 6. Both perturbations have roughly the same performance (around -5.3), VFI a slightly better -6.4, while Chebyshev polynomials do fantastically well at -10.4 (the average loss of welfare is $1 for each $500 billions). But even an approximation with an average error of $1 for each $200,000, such as the one implied by third-order perturbation must suffice for most relevant applications.

We repeat our exercise for the extreme calibration. Figure 6 displays the results for the extreme case. Again, we have changed the capital interval to make it representative of the behavior of the model in the ergodic distribution. Now, perturbations worsen more as we get further away from the deterministic steady state. However, in the relevant range of values of capital of [6, 17], we still have Euler equation errors smaller than -3 and, hence, probably small enough for most applications of interest. The performance of VFI deteriorates around one order of magnitude with respect to our benchmark calibration. Chebyshev polynomials suffer more in relative terms (they started at a quite outstanding accuracy level), but they still deliver the smallest errors in nearly all the relevant range of capital.

Table 7 reports maximum Euler equation errors and their integrals. The maximum Euler equation error is large for perturbation methods while it is rather small using Chebyshev polynomials. However, given the very large range of capital used in the computation, this maximum Euler error provides a too negative view of accuracy. We find the integral of the Euler equation error to be much more instructive. With a second-order perturbation, we have -4.02 and with a third-order perturbation we have -4.12. To evaluate this number, remember that we have extremely high risk aversion and large

\[
\text{since this is an object that can only be found by simulation. We use the ergodic simulation generated by VFI, which slightly favors this method over the other ones. However, we checked that the results are robust to using the ergodic distributions coming from the other methods.} \]
productivity shocks. Even in this challenging environment, perturbations delivers a high degree of accuracy. VFI does not display a big loss of precision compared to the benchmark case. On the other hand, Chebyshev polynomials deteriorate somewhat, but the accuracy it delivers it is still of $1$ out of each $1,000,000$ spent.

4.5.4 Robustness: Changing the EIS and Changing the Perturbation Point

In the results we reported above, we kept the EIS equal to 0.5, a conventional value in the literature, while we modified the risk aversion and the volatility of productivity shocks. However, since some researchers prefer higher values of the EIS (see, for instance, Bansal and Yaron, 2004, a paper that we have used to motivate our investigation), we also computed our model with $\psi = 1.5$. Basically our results were unchanged. To save on space, we concentrate only on the Euler equation errors (decision rules and simulation paths are available upon request). In table 8, we report the maxima of the Euler equation errors and their integrals with respect to the ergodic distribution. The relative size and values of the entries of this table are quite similar to the entries in table 6 (except, partially, VFI that performs a bit better).

Table 9 repeats the same exercise for the extreme calibration with high risk aversion and high volatility of productivity shocks. Again, the entries on the table are very close to the ones in table 7 (and now, VFI does not perform better than when $\psi = 0.5$).

As a final robustness test, we computed the perturbations not around the deterministic steady state (as we did in the main text), but around a point close to the mode of the ergodic distribution of capital. This strategy, if perhaps difficult to implement because of the need to compute the mode of the ergodic distribution,\(^6\) could deliver better accuracy because we approximate

\(^6\)For example, the algorithm of finding a perturbation around the steady state, simulate from it, find a second perturbation around the model of the implied ergodic simulation,
4.5. NUMERICAL RESULTS

the value function and decision rules in a region where the model spends more time. As we suspected, we found only trivial improvements in terms of accuracy (for instance, the Euler equation errors improved by less than 1 percent). Moreover, expanding at a point different from the deterministic steady state has the disadvantage that the theorems that ensure the convergence of the Taylor approximation might fail (see theorem 6 in Jin and Judd, 2002).

4.5.5 Implementation and Computing Time

We briefly discuss implementation and computing time. For the benchmark calibration, second-order perturbation and third-order perturbation algorithms take only 0.02 second and 0.05 second, respectively, in a 3.3GHz Intel PC with Windows 7 (the reference computer for all times below), and it is simple to implement: 664 lines of code in Fortran 95 for second order and 1133 lines of code for third order, plus in both cases, the analytical derivatives of the equilibrium conditions that Fortran 95 borrows from a code written in Mathematica 6.\(^7\) The code that generates the analytic derivatives has between 150 to 210 lines, although Mathematica is much less verbose. While the number of lines doubles in the third order, the complexity in terms of coding does not increase much: the extra lines are mainly from declaring external functions and reading and assigning values to the perturbation coefficients. An interesting observation is that we only need to take the analytic derivatives once, since they are expressed in terms of parameters and not in terms of parameter values. This allows Fortran to evaluate the analytic derivatives extremely fast for new combinations of parameter values. This

---

\(^7\)We use lines of code as a proxy for the complexity of implementation. We do not count comment lines.
advantage of perturbation is particularly relevant when we need to solve the model repeatedly for many different parameter values, for example, when we are estimating the model. For completeness, the second-order perturbation was also run in Dynare (although we had to use version 4.0, which computes analytic derivatives, instead of previous versions, which use numerical derivatives that are not accurate enough for perturbation). This run was a double-check of the code and a test of the feasibility of using off-the-shelf software to solve DSGE models with recursive preferences.

The projection algorithm takes around 300 seconds, but it requires a good initial guess for the solution of the system of equations. Finding the initial guess for some combination of parameter values proved to be challenging. The code is 652 lines of Fortran 95. Finally, the VFI code is 707 lines of Fortran 95, but it takes about ten hours to run.

4.6 Conclusions

In this paper, we have compared different solution methods for DSGE models with recursive preferences and SV. We evaluated the different algorithms based on accuracy, speed, and programming burden. We learned that all of the most promising methods (perturbation, projection, and VFI) do a fair job in terms of accuracy. We were surprised by how well simple second-order and third-order perturbations perform even for fairly non-linear problems. We were impressed by how accurate Chebyshev polynomials can be. However, their computational cost was higher and we are concerned about the curse of dimensionality. In any case, it seems clear to us that, when accuracy is the key consideration, Chebyshev polynomials are the way to go. Finally, we were disappointed by VFI since even with 125,000 points in the grid, it only did marginally better than perturbation and it performed much worse than Chebyshev polynomials in our benchmark calibration. This suggests that unless there are compelling reasons such as non-differentiabilities or
non-convexities in the model, we better avoid VFI.

A theme we have not developed in this paper is the possibility of interplay among different solution methods. For instance, we can extremely easily compute a second-order approximation to the value function and use it as an initial guess for VFI. This second-order approximation is such a good guess that VFI will converge in few iterations. We verified this idea in non-reported experiments, where VFI took one-tenth of the time to converge once we used the second-order approximation to the value function as the initial guess. This approach may even work when the true value function is not differentiable at some points or has jumps, since the only goal of perturbation is to provide a good starting point, not a theoretically sound approximation. This algorithm may be particularly useful in problems with many state variables. More research in this type of hybrid method is a natural extension of our work.

We close the paper by pointing out that recursive preferences are only one example of a large class of non-standard preferences that have received much attention by theorists and applied researchers over the last years (see Backus et al., 2004). Having fast and reliable solution methods for this class of new preferences will help researchers sort out which of these preferences deserve further attention and to derive empirical implications. Thus, this paper is a first step in the task of learning how to compute DSGE models with non-standard preferences.

Bibliography


A Appendix

In this appendix, we present the steady state of the model and the alternative perturbation approach, the value function perturbation (VFP).

A.1 Steady State of the Model

To solve the system:

\[
\begin{align*}
V_{ss} &= c_{ss} (1 - l_{ss})^{1-v} \\
(\zeta k_{ss}^{-1})^{1-\zeta} + 1 - \delta &= 1/\beta \\
1 - \nu \frac{c_{ss}}{\nu (1 - l_{ss})} &= (1 - \zeta) k_{ss}^{\zeta} l_{ss}^{-\zeta} \\
m_{ss} P_{ss}^{f} &= 1/\beta \\
c_{ss} + i_{ss} &= k_{ss}^{\zeta} l_{ss}^{-\zeta} \\
i_{ss} &= \delta k_{ss}
\end{align*}
\]

note first that:

\[
k_{ss}^{\zeta} l_{ss}^{-\zeta} = \Omega
\]

Now, from the leisure-consumption condition:

\[
\frac{c_{ss}}{1 - l_{ss}} = \frac{\nu}{1 - \nu} (1 - \zeta) \Omega^{\zeta} = \Phi \Rightarrow c_{ss} = \Phi (1 - l_{ss})
\]

Then:

\[
c_{ss} + \delta k_{ss} = k_{ss}^{\zeta} l_{ss}^{-\zeta} = \Omega^{\zeta} l_{ss} \Rightarrow c_{ss} = (\Omega^{\zeta} - \delta) l_{ss}
\]

and:

\[
\Phi (1 - l_{ss}) = (\Omega^{\zeta} - \delta \Omega) l_{ss} \Rightarrow \\
l_{ss} = \frac{\Phi}{\Omega^{\zeta} - \delta \Omega + \Phi}
\]
from which we can find $V_{ss}$ and $i_{ss}$.

### A.2 Value Function Perturbation (VFP)

We mentioned in the main text that instead of perturbing the equilibrium conditions of the model, we could directly perturb the value function in what we called value function perturbation (VFP). To undertake the VFP, we write the value function as:

$$V(k_t, z_t, \sigma_t; \chi) = \max_{c_t, l_t} \left[ (1 - \beta) \left( c_t^\nu (1 - l_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta E_t V^{1-\gamma} (k_{t+1}, z_{t+1}, \sigma_{t+1}; \chi) \right]^{\frac{\theta}{1-\gamma}}$$

To find a second-order approximation to the value function, we take derivatives of the value function with respect to controls $(c_t, l_t)$, states $(k_t, z_t, \sigma_t)$, and the perturbation parameter $\chi$. We collect these 6 equations, together with the resource constraint, the value function itself, and the exogenous processes in a system:

$$\tilde{F}(k_t, z_t, \chi) = 0$$

where the hat over $F$ emphasizes that now we are dealing with a slightly different set of equations than the $F$ in the main text.

After solving for the steady state of this system, we take derivatives of the function $\tilde{F}$ with respect to $k_t, z_t, \sigma_t$, and $\chi$:

$$\tilde{F}_i(k_{ss}, \sigma_{ss}; 0) = 0 \text{ for } i = \{1, 2, 3, 4\}$$

and we solve for the unknown coefficients. This solution will give us a second-order approximation of the value function but only a first-order approximation of the decision rules. By repeating these steps $n$ times, we can obtain the $n + 1$-order approximation of the value function and the $n$-order approximation of the decision rules. It is straightforward to check that the coefficients
obtained by ECP and VFP are the same. Thus, the choice for one approach or the other should be dictated by expediency.
A. APPENDIX

A.3 Tables and Figures

Table 1: Calibrated Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\beta$</th>
<th>$\nu$</th>
<th>$\zeta$</th>
<th>$\delta$</th>
<th>$\lambda$</th>
<th>$\log \sigma$</th>
<th>$\rho$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.991</td>
<td>0.357</td>
<td>0.3</td>
<td>0.0196</td>
<td>0.95</td>
<td>0.007</td>
<td>0.9</td>
<td>0.06</td>
</tr>
</tbody>
</table>

Table 2: Business Cycle Statistics - Benchmark Calibration

<table>
<thead>
<tr>
<th></th>
<th>$c$</th>
<th>$y$</th>
<th>$i$</th>
<th>$R^f(%)$</th>
<th>$R^k(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Second-Order Perturbation</td>
<td>0.7253</td>
<td>0.9128</td>
<td>0.1873</td>
<td>0.9070</td>
<td>0.9078</td>
</tr>
<tr>
<td>Third-Order Perturbation</td>
<td>0.7257</td>
<td>0.9133</td>
<td>0.1875</td>
<td>0.9062</td>
<td>0.9069</td>
</tr>
<tr>
<td>Chebyshev Polynomial</td>
<td>0.7256</td>
<td>0.9130</td>
<td>0.1875</td>
<td>0.9063</td>
<td>0.9066</td>
</tr>
<tr>
<td>Value Function Iteration</td>
<td>0.7256</td>
<td>0.9130</td>
<td>0.1875</td>
<td>0.9063</td>
<td>0.9066</td>
</tr>
<tr>
<td>Variance (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Second-Order Perturbation</td>
<td>0.0331</td>
<td>0.1084</td>
<td>0.0293</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>Third-Order Perturbation</td>
<td>0.0330</td>
<td>0.1079</td>
<td>0.0288</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>Chebyshev Polynomial</td>
<td>0.0347</td>
<td>0.1117</td>
<td>0.0313</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>Value Function Iteration</td>
<td>0.0347</td>
<td>0.1117</td>
<td>0.0313</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Table 3: Welfare Costs of Business Cycle - Benchmark Calibration

<table>
<thead>
<tr>
<th>2nd-Order Pert.</th>
<th>3rd-Order Pert.</th>
<th>Chebyshev</th>
<th>Value Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.0864e(-5)</td>
<td>-2.0864e(-5)</td>
<td>-3.2849e(-5)</td>
<td>-3.2849e(-5)</td>
</tr>
</tbody>
</table>
Table 4: Business Cycle Statistics - Extreme Calibration

<table>
<thead>
<tr>
<th>Mean</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c$</td>
<td>$y$</td>
<td>$i$</td>
<td>$R_f$</td>
</tr>
<tr>
<td>Second-Order Perturbation</td>
<td>0.7338</td>
<td>0.9297</td>
<td>0.1950</td>
<td>0.8432</td>
</tr>
<tr>
<td>Third-Order Perturbation</td>
<td>0.7344</td>
<td>0.9311</td>
<td>0.1955</td>
<td>0.8416</td>
</tr>
<tr>
<td>Chebyshev Polynomial</td>
<td>0.7359</td>
<td>0.9329</td>
<td>0.1970</td>
<td>0.8331</td>
</tr>
<tr>
<td>Value Function Iteration</td>
<td>0.7359</td>
<td>0.9329</td>
<td>0.1970</td>
<td>0.8352</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variance (%)</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Second-Order Perturbation</td>
<td>0.2956</td>
<td>1.0575</td>
<td>0.2718</td>
<td>0.0004</td>
</tr>
<tr>
<td>Third-Order Perturbation</td>
<td>0.3634</td>
<td>1.2178</td>
<td>0.3113</td>
<td>0.0004</td>
</tr>
<tr>
<td>Chebyshev Polynomial</td>
<td>0.3413</td>
<td>1.1523</td>
<td>0.3425</td>
<td>0.0005</td>
</tr>
<tr>
<td>Value Function Iteration</td>
<td>0.3414</td>
<td>1.1528</td>
<td>0.3427</td>
<td>0.0005</td>
</tr>
</tbody>
</table>

Table 5: Welfare Costs of Business Cycle - Extreme Calibration

<table>
<thead>
<tr>
<th></th>
<th>2nd-Order Pert.</th>
<th>3rd-Order Pert.</th>
<th>Chebyshev</th>
<th>Value Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1278e-2</td>
<td>1.1278e-2</td>
<td>1.2855e-2</td>
<td>1.2838e-2</td>
<td></td>
</tr>
</tbody>
</table>

Table 6 Euler errors - Benchmark Calibration

<table>
<thead>
<tr>
<th></th>
<th>Max Euler Error</th>
<th>Integral of the Euler Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second-Order Perturbation</td>
<td>-2.6294</td>
<td>-5.2350</td>
</tr>
<tr>
<td>Third-Order Perturbation</td>
<td>-2.7437</td>
<td>-5.3164</td>
</tr>
<tr>
<td>Chebyshev Polynomial</td>
<td>-9.7919</td>
<td>-10.4034</td>
</tr>
<tr>
<td>Value Function Iteration</td>
<td>-3.0848</td>
<td>-6.4039</td>
</tr>
</tbody>
</table>
### Table 7 Euler errors - Extreme Calibration

<table>
<thead>
<tr>
<th>Method</th>
<th>Max Euler Error</th>
<th>Integral of the Euler Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second-Order Perturbation</td>
<td>-1.5188</td>
<td>-4.0195</td>
</tr>
<tr>
<td>Third-Order Perturbation</td>
<td>-1.6698</td>
<td>-4.1189</td>
</tr>
<tr>
<td>Chebyshev Polynomial</td>
<td>-4.8979</td>
<td>-5.9339</td>
</tr>
<tr>
<td>Value Function Iteration</td>
<td>-2.5186</td>
<td>-6.2870</td>
</tr>
</tbody>
</table>

### Table 8 Euler errors - Benchmark Calibration with $\psi = 1.5$.

<table>
<thead>
<tr>
<th>Method</th>
<th>Max Euler Error</th>
<th>Integral of the Euler Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second-Order Perturbation</td>
<td>-2.5525</td>
<td>-5.0811</td>
</tr>
<tr>
<td>Third-Order Perturbation</td>
<td>-3.0184</td>
<td>-5.1395</td>
</tr>
<tr>
<td>Chebyshev Polynomial</td>
<td>-9.5192</td>
<td>-10.1104</td>
</tr>
<tr>
<td>Value Function Iteration</td>
<td>-3.4633</td>
<td>-6.5078</td>
</tr>
</tbody>
</table>

### Table 9 Euler errors - Extreme Calibration with $\psi = 1.5$.

<table>
<thead>
<tr>
<th>Method</th>
<th>Max Euler Error</th>
<th>Integral of the Euler Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second-Order Perturbation</td>
<td>-1.5835</td>
<td>-3.8473</td>
</tr>
<tr>
<td>Third-Order Perturbation</td>
<td>-1.6406</td>
<td>-3.9917</td>
</tr>
<tr>
<td>Chebyshev Polynomial</td>
<td>-4.6367</td>
<td>-5.6323</td>
</tr>
<tr>
<td>Value Function Iteration</td>
<td>-2.3661</td>
<td>-5.5882</td>
</tr>
</tbody>
</table>
Figure 1: Decision Rules and Value Function, Benchmark Calibration
Figure 2: Decision Rules and Value Function, Extreme Calibration
Figure 3: Densities, Benchmark Calibration
Figure 4: Densities, Extreme Calibration
Figure 5: Euler Equation Error, Benchmark Calibration
Figure 6: Euler Equation Error, Extreme Calibration
Chapter 5

Business Cycle Accounting and Misspecified DSGE Models*

5.1 Introduction

Over the past decade, there has been a marked increase in the role of dynamic stochastic general equilibrium (DSGE) models in policy institutions. The seminal work of Smets and Wouters (2003) is regarded by many as a “proof of concept” (Sims, 2008) that medium-scale DSGE models estimated using Bayesian methods can be useful tools for policy analysis.1 Smets and Wouters (2003) showed that models of this type could deliver a reasonable forecast performance as well as the "story telling" capabilities that flow from explicit assumptions about the optimization decisions of economic agents.2 Indeed, a number of central banks have recently developed operational forecast models based on this blueprint. Prominent recent examples include the RAMSES

---

*This chapter is co-authored with Richard Harrison. The views in this paper are those of the authors and do not necessarily reflect those of the Bank of England’s Monetary Policy Committee.

1For an overview of the estimation of DSGE models, see Fernandez-Villaverde (2010) and Schorfheide (2011).

2Naturally, both the forecasting and story telling abilities can be questioned. See, for example, Sims (2007).
model developed at Sveriges Riksbank and the NAWM model of the ECB.\(^3\)

Though RAMSES and NAWM follow the approach pioneered by Smets and Wouters (2003, 2007), they dwarf them in scale. While the Smets and Wouters models are fit to seven data series, RAMSES and NAWM are designed to explain the behavior of fifteen and twenty-one series, respectively. One reason why operational central bank models are larger than their academic counterparts surely stems from policymakers’ desire to have detailed and comprehensive discussions about a large number of shocks and transmission channels.

But all models, notwithstanding how large, are misspecified. For example, to our knowledge, none of the DSGE models in operational use at central banks contain an explicit modeling of financial frictions or banking. This is not to say that models with such features do not exist. Indeed, research on these issues is currently a very fertile area. And one response to the observation that operational models exclude some channels and mechanisms of interest is to expand them accordingly. Naturally, there are some difficulties associated with this approach: if the model is to be estimated, then computational considerations place a (practical) upper bound on the number of observable variables; and larger models are inherently harder to understand and explain to busy policymakers. But even if this strategy is a desirable long-term objective, in the short run it is possible that the economic issues relevant for policy discussions develop more quickly than the operational forecast models used to support those discussions. This suggests that putting too much faith in one model may be misguided and often leads policy institutions to use a variety of models in the preparation of their forecast and policy analysis.

In this paper, we consider how insights from other DSGE models (and po-

\(^3\)RAMSES is discussed in detail in Adolfson et al. (2007). The NAWM is introduced in Christoffel et al. (2008). Other examples are the Norges Bank’s NEMO (Brubakk et al., 2006) and the Federal Reserve Board’s FRB/EDO model (Edge et al., 2007; Chung et al., 2010).
potentially less structural types of models such as SVARs) can be used to trace out the implications of "missing channels" in a baseline estimated DSGE model used for forecast and policy analysis. Specifically, we set out some ideas for how the insights from the business cycle accounting (BCA) methodology may be applied to the issue of misspecification in DSGE models.

The paper is structured as follows. In Section 5.2, we provide a sketch of our argument that the insights of BCA can be applied to the issue of misspecification in DSGE models. Thus, we place our ideas in the context of the existing literature. In Section 5.3, we illustrate the approach using two simple examples. These examples consider cases where a policymaker has access to a misspecified model of the economy. Our first example is one where oil prices affect the supply-side of the economy, but the policymaker’s model does not include a role for oil. Our second example is one where house prices have a financial accelerator effect on demand, but the policymaker’s model does not include house prices or any mechanisms through which they may play an important role.

In Section 5.4, we present an empirical example of the approach. We use the DSGE model of Smets and Wouters (2007) to construct baseline projections for key macro variables and then investigate how our approach can be used to adjust those projections in light of alternative scenarios for the future path of house prices (which are not included in the baseline model). Section 5.5 concludes the paper.

5.2 A Sketch of the Idea

This section focuses on providing the logic of our idea for modifying a baseline DSGE model to incorporate "missing channels". Here, we give a summary of our argument before looking at its individual components in more detail.

DSGE models are seen as useful tools for analyzing data and producing forecasts in terms of underlying "structural shocks" that hit the economy.
Generally speaking, DSGE models contain a fairly large number of shocks: usually the number of structural shocks is the same as the number of observable variables that the model is asked to explain. The explicit mapping from structural shocks to endogenous variables is seen as a key strength of the approach. However, in order to identify that mapping for a model containing a reasonably large number of variables, assumptions must be imposed on the behavior of the structural shock processes. Typically, the innovations in the exogenous processes are assumed to be independent. Thus, for example, a shock to the total factor productivity (TFP) process will be uncorrelated with shocks to firms’ mark-ups. The stochastic processes assumed for the shocks are typically specified to be very parsimonious. So it is typical to assume that the TFP and markup shocks each follow AR(1) processes (so that lagged TFP does not affect the current markup and vice versa).

The identifying assumptions placed on the structural shock processes facilitate the estimation of the model. Allowing a general cross-correlation structure between shocks would introduce a large number of additional parameters to be estimated. Moreover, if we believe that the DSGE model is a good description of how the economy actually works, then orthogonality of the innovations to the structural shocks should be a good assumption. The disturbances driving the economy are fundamental shocks that should, by definition, be independent of one another. Finally, the orthogonality assumption is useful for story telling. When we examine the effects of a TFP shock on our model, it is convenient to trace the effects from the shock to TFP and then onto the endogenous variables in the model. This way of examining the impulse response function is greatly facilitated by the assumption that each structural shock follows an independent AR process. Otherwise, we must attempt to disentangle the effects on output from the propagation of the TFP innovation through other structural shocks in the model.

\footnote{Naturally, we can estimate stochastically singular DSGE models by the appropriate inclusion of measurement error. See Ireland (2004) for a discussion.}
The standard DSGE approach has some clear advantages. But, as noted above, it is likely that these advantages may only be fully realized if the model is well specified. Our view is that any DSGE model will be misspecified. The BCA literature takes model misspecification very seriously. For these authors, models are laboratories that can be used to test theories about how the economy works, but they should not be expected to replicate the behavior of all variables.

The BCA literature contains two important results that are useful for our purposes. First, a number of papers have demonstrated that it is possible to represent a wide variety of DSGE models in terms of a very simple prototype real business cycle (RBC) model that contains stochastic disturbances (or wedges) to the optimality conditions of the model. The wedges can be considered in terms of distortionary taxes that push equilibrium prices and allocations away from their undistorted values. Specifically, Chari et al. (2007) show that "detailed economy" models with input-financing, financial accelerators, sticky prices and sticky wages can all be mapped into a prototype RBC model with time-varying wedges. The nature of this mapping is as follows: it is possible to find sequences of wedges in the prototype model such that allocations in that model are identical to those in the detailed model. The second result is that the sequences of wedges in the prototype RBC model required to match the allocations in a detailed model are typically correlated. So the wedges in the BCA approach cannot be interpreted as structural shocks.\(^5\)

The methodological differences between proponents of the BCA and DSGE approaches reflect some fundamental differences in the view of how models should be used. The approach outlined in this note attempts to take both views into account. We argue that a baseline DSGE model is a useful way of placing some identification on the shocks driving the data. But we

\(^5\)See Sustek (forthcoming) for a discussion of the relationship between DSGE and BCA approaches.
also argue that we need to be careful when interpreting the results of this approach. Specifically, because our model identifies, say, a significant wage markup shock as important in particular episodes, it does not mean that we should believe that sharp quarter-to-quarter changes in the competitiveness of the labor market are the key drivers of those episodes. As pointed out by Chari et al. (2009), standard New Keynesian DSGE models (such as that of Smets and Wouters, 2007) imply that such shocks play an important role in business cycle fluctuations. But when interpreted literally, the implied quarter-to-quarter changes in labor market competitiveness are truly incredible. This immediately tells us that the wage markup shock identified by the model cannot literally be interpreted as changes in the elasticity of labor demand. Chari et al. (2009) show that it is impossible to disentangle the effects of movements in this parameter from competing labor market assumptions. They argue persuasively that this implies that under these conditions, it is hard to argue that these models can be used reliably for welfare analysis.

While we accept the sentiment of the above arguments, we do not agree with the implication that DSGE models are inherently useless for forecast or policy analysis. Instead, a pragmatic approach suggests that we can use DSGE models as a starting point for a deeper economic enquiry. That is, we need to probe more deeply behind the "labels" placed by our models on the important shocks driving the data to uncover more fundamental stories.

This pragmatic approach leads us to attempt to combine some important strengths of the DSGE and BCA methodologies. We argue that a baseline DSGE model can be a convenient starting point for the analysis. But we recognize that a key insight from BCA analysis is that transmission channels that are not captured in the baseline model may appear as correlated shocks to that model. We argue that it may be possible to map from missing channels by applying the BCA insights to the baseline DSGE model. The key idea is to introduce a proxy variable that captures the effects of a missing channel and relate the innovations to this proxy variable to a (small) set of atheoretical
"factors". We then allow these factors to feed into the structural shocks of
the model to create correlated movements in those shocks.

There has been a number of attempts to address the issue that the number
of variables matched by operational DSGE models is typically somewhat
smaller than the range of variables that enter policy and forecast discussions.
Boivin and Giannoni (2006) fit a DSGE model to a large number of variables
using a dynamic factor approach where the state vector of the DSGE model
is treated as the latent factor. Large numbers of measurement equations are
then constructed to map out the relationship between the DSGE state vector
and the observed data. This approach has two attractive properties. First, it
allows the forecaster to use a large number of variables to help identify the
latent state vector, which should improve the forecast performance. Second,
it allows the forecaster to produce a much broader array of projections, all
consistent with the underlying DSGE projection, to feed into the forecast and
policy discussion. Schorfheide et al. (2008) note that the Boivin and Giannoni
(2006) approach is computationally demanding, which reduces its practical
appeal. They propose a similar method for producing recursive projections
of non-modeled variables, based on the state vector of the DSGE model.  

These approaches share some similarities, at least in spirit, with the
"core/non-core" approach used in the Bank of England Quarterly Model
(BEQM). The core/non-core (CNC) approach combines a "core" DSGE
model with a set of VECM-style "non-core" equations. The non-core equations
are considered as measurement equations that also include a role for
variables that are not modeled inside the core, as explained by Alvarez-Lois
et al. (2008). As noted by Alvarez-Lois et al. (2008), one drawback of the
CNC approach is that the projections of the non-core variables do not form
any part of the information set used by agents in the core model. This makes
it difficult to produce fully model-consistent projections for scenarios that

---

6 The cost of the increased tractability is that the additional variables are not used to
improve the identification of the state vector.

7 See Harrison et al. (2005).
involve an alternative path for a non-core variable.

This leads Alvarez-Lois et al. (2008) to suggest that additional *ad hoc* terms be inserted into the equations describing the structural shocks of a DSGE model, which has the effect of making the additional variables part of the state vector of the model. This means that changes in the paths of these variables will be factored into the decisions of DSGE model agents, thus making it easier to produce model-consistent scenarios. We extend their analysis by analyzing how similar modifications to DSGE models may perform in practice as well as more carefully considering the issues associated with estimating such models. We note that there are several examples of DSGE models that incorporate a flavor of these ideas. One is the inclusion of lagged exchange rate terms in the UIP condition in RAMSES (see Sims (2007) for a discussion). Other examples are Smets and Wouters (2007) (who assume government expenditure and TFP shocks to be correlated) and Juillard et al. (2006) (who allow for a correlation between investment and marginal utility shocks).

In what follows, we provide a more formal account of the approach. We start by reviewing the DSGE and BCA approaches before describing a combination of the two.

### 5.2.1 DSGE and BCA Approaches

We start with the state space representation of the equilibrium of a DSGE model

\[
\begin{align*}
    z_t &= DS_t, \\
    S_t &= GS_{t-1} + H\varepsilon_t,
\end{align*}
\]

where

\[
S_t \equiv \begin{bmatrix} Z_t \\ s_t \end{bmatrix}.
\]
and all variables are measured as log-deviations from steady state. Here, $S$ is
the state vector collecting together the forcing processes, or structural shocks,
s, and the predetermined endogenous variables, $Z$. We denote by $z$ the vector
of non-predetermined endogenous variables and $\varepsilon$ is a vector of orthogonal
structural shocks with a diagonal covariance matrix $Q$. Matrices $D$, $G$ and
$H$ are functions of the DSGE parameter vector $\theta$ (though we suppress this
dependence for notational convenience).

The structural shocks are typically modeled as a VAR:

$$s_t = Bs_{t-1} + C\varepsilon_t,$$

where $B$ and $C$ are usually diagonal. As noted in Section 5.2, the assumptions
that $B$ is diagonal and that the shocks $\varepsilon_t$ are orthogonal have two key advantages. First, these assumptions reduce the number of parameters in the model. Second, they add to the ability of the model to tell coherent stories. Because the structural shocks are given an economic interpretation, it is important that innovations to them are orthogonal. Orthogonality makes it easier to trace through the effects of an exogenous impulse through the structural shock processes and onto the endogenous variables in the model.

The BCA methodology takes a different approach. The idea is that the economy can be approximated by a simple DSGE model where the forcing processes represent distortions to the equilibrium conditions of the model. Very few restrictions are placed on the evolution of these distortions – or wedges – because they are not given a particular economic interpretation. So, for example, the forcing processes might follow a VAR:

$$s_t = Vs_{t-1} + W\varepsilon_t,$$

where $V$ and $W$ are typically subject to only minimal restrictions (eg stability).

Naturally, the BCA approach usually confines the attention to a proto-
typical RBC model, which has implications for the precise structure of the DSGE state space. One reason, as noted in Section 5.2, is that BCA researchers wish to impose a minimal structure on the economic behavior of agents in the model, preferring to analyze the frictions that drive the data in terms of the unrestricted VAR process that drives the wedges. Naturally, a corollary is that using an unrestricted VAR to describe the evolution of the wedges implies that the dimension of the parameter space is very large and thus, successful estimation requires a relatively small model.

5.2.2 DSGE Meets BCA?

Our proposal is to combine elements of the DSGE and BCA approaches. We aim at striking a balance between the parsimony and story-telling power of a well-specified DSGE model with the BCA insight that, in general, such models will not be well-specified and missing channels will tend to manifest themselves as correlated shocks. Our approach can be seen as a hybrid of the two approaches in the sense that we assume the structural shocks to be the sum of two components. The first component is a traditional DSGE structural shock. Innovations to this component can be traced through the model and the story of how that shock affects the endogenous variables can be constructed as usual. The second component of each structural shock consists of a weighted sum of a relatively small number of “atetheoretical” factors that may contain a rich cross-correlation structure. These factors are designed to pick up the effects of a missing channel that is not part of the DSGE model. They can be linked to the missing channel by means of an equation that identifies shocks to a proxy variable (that is intended to capture some of the effects of the missing channel). The innovations to the proxy variable are assumed to drive the atetheoretical factors.

Specifically, suppose that the forcing processes are given by:
where – as described above – the forcing processes are decomposed into two components: a "true" structural shock ($\tilde{s}$) which is modeled in the conventional DSGE manner; and a set of atheoretical factors ($F$) which are modeled using a flexible VAR specification. We denote missing channel proxies as $m$.

The logic is that we use observations of $m$ to identify the shocks driving that process, $u$ and then model the way that these shocks affect the factors $F$ that drive the structural shock processes. If the number of atheoretical factors is kept small relative to the number of structural shocks, then the number of additional parameters involved in the new specification of the forcing process will be far smaller than the number of additional parameters that would be needed to incorporate a fully flexible (i.e. unrestricted) VAR for the structural shocks, $s$.

Naturally, there are many potential variants of the scheme proposed above. For example, while the above approach makes sense if we can think of $m_t$ as purely exogenous, an alternative specification (allowing for the potential endogeneity of $m_t$) would be:

\[
\begin{align*}
    s_t &= \tilde{s}_t + \Lambda F_t, \\
    \tilde{s}_t &= B\tilde{s}_{t-1} + C\varepsilon_t, \\
    F_t &= \Phi F_{t-1} + \Psi u_t, \\
    m_t &= \Xi F_t + u_t,
\end{align*}
\]

where in this interpretation we view the missing channel variable(s) $m_t$ as a
proxy for a deeper missing process. Note that in this representation, innovations to the unexplained component of the missing channel proxy \((u)\) do not have any effects on the model through the forcing processes (in contrast to 5.1). So this approach is similar to that of Boivin and Giannoni (2006). In this case, analyzing the effects of changes in \(m\) on the forcing processes (and hence endogenous variables of interest) would need to be done through shocks to the factors \(F\). One possibility of investigating the implications of a particular profile for the missing channel proxy would be to back out the most likely sequence of shocks to the underlying factors that would deliver the desired path.

However, it is likely that both of these approaches are misspecified in the sense that the true process for a missing channel proxy is also likely to depend on the endogenous state variables:

\[
m_t = \Xi F_t + \Theta Z_{t-1} + u_t,
\]

but accounting for this dependence would involve introducing a large number of additional parameters (since \(\Theta\) is an \(n_m \times n_s\) matrix, where \(n_m\) is the number of missing channel proxies and \(n_s\) is the number of states).\(^8\)

### 5.2.3 How the Idea Can be Applied to Existing Models

In this section, we suppose that we start with a specification of a DSGE model that does not incorporate any missing channels and consider how they can be added algebraically. Suppose that the model is written as:

\[
M_1 E_t X_{t+1} = M_0 X_t + M_s s_t, \tag{5.2}
\]

\[
s_t = B s_{t-1} + \varepsilon_t,
\]

\(^8\)This point is analogous to the argument of Baeurle and Burren (2011) that BCA approaches should condition the process for wedges on the state variables of the so-called prototype economy.
where \( s_t \) are the structural shock processes. To augment the model in the way suggested above, we need to introduce additional forcing processes. The structural equations of the model can be left intact, but the evolution of the shocks can be modified to:

\[
\begin{align*}
    s_t &= \tilde{s}_t + \Lambda F_t, \\
    \tilde{s}_t &= B\tilde{s}_{t-1} + \varepsilon_t, \\
    F_t &= \Phi F_{t-1} + \Psi u_t, \\
    m_t &= Jm_{t-1} + u_t,
\end{align*}
\]

where we may stack all this information into a new vector \( R_t \equiv [s'_t \; \tilde{s}'_t \; F'_t \; m'_t]' \) which has dynamics:

\[
R_t = \begin{bmatrix}
0 & B & \Lambda \Phi & 0 \\
0 & B & 0 & 0 \\
0 & 0 & \Phi & 0 \\
0 & 0 & 0 & J
\end{bmatrix} R_{t-1} + \begin{bmatrix}
\varepsilon_t + \Lambda \Psi u_t \\
\varepsilon_t \\
\Psi u_t \\
u_t
\end{bmatrix}
\]

or

\[
R_t = \Pi R_{t-1} + e_t,
\]

where the covariance matrix of the composite shock is:

\[
E [e_t e'_t] = \begin{bmatrix}
\Sigma_\varepsilon + \Lambda \Psi \Sigma_u \Psi' \Lambda' & - & - & - \\
\Sigma_\varepsilon & \Sigma_\varepsilon & - & - \\
\Lambda \Psi \Sigma_u \Psi' \Lambda' & 0 & \Psi \Sigma_u \Psi' & - \\
\Sigma_u \Psi' \Lambda' & 0 & \Sigma_u \Psi' & \Sigma_u
\end{bmatrix}.
\]

Inserting this description of the augmented state vector into the structural
equations of the model gives a new system:

\[
M_1 E_t X_{t+1} = M_0 X_t + M_R R_t, \\
R_t = \Pi R_{t-1} + e_t,
\]

where

\[
M_R = \begin{bmatrix} M_s & 0 \end{bmatrix}.
\]

This simple algebra enables us to take a model in the form (5.2) and apply additional assumptions about the forcing processes to deliver an augmented model of the form (5.3). Since the structure of (5.3) is essentially the same as the structure of (5.2), we can solve the model using standard methods. Indeed, since many solution approaches based on eigenvalue methods do not require the shock processes to be specified, it will often be possible to directly augment the state space representation of the model.

5.3 Two Simple Examples

In this section, we provide two simple examples of the approach outlined in Section 5.2. The setting we wish to investigate is the following. We suppose that a policymaker has a misspecified DSGE model that excludes a sector or transmission channel that may be important for the determination of macroeconomic dynamics.

In Section 5.3.1, we consider an example where oil prices affect the supply side of the economy, but the policymaker’s model does not include any role for oil. In Section 5.3.2, we consider another case where house prices have a financial accelerator effect on demand, but the policymaker’s model does not include house prices or any mechanisms through which they may play an important role.

In each case, the task is to augment a baseline DSGE model in an attempt to capture the effects of the missing channel. We focus on the task
of approximating the marginal responses of the economy to changes in the missing variable. That is, we examine whether our technique can be used to approximate the impulse response to a shock that is transmitted via the missing channel. In each case, we suppose that the policymaker has access to a very long data set generated by the true model. This allows us to eliminate small sample estimation issues, though in practice these are likely to be important.

We conclude with a discussion of the technique and ways in which it can be modified for empirical applications, which is the subject of Section 5.4.

5.3.1 Missing Oil Prices

Here we assume that the policymaker’s model is misspecified in the sense that it omits the effects of oil price shocks on economic activity. The policymaker has data for oil prices and wants to analyze the implications of an oil price shock for the endogenous variables in the misspecified model. The task is to attempt to compute the impulse response to an oil price shock in a model without oil.

We proceed in the following steps:

1. Specify the data generating process (Section 5.3.1.1)

2. Specify the policymaker’s (misspecified) model and fit it to the data generating process (Section 5.3.1.2)

3. Augment the policymaker’s model to try to account for the missing channel (Section 5.3.1.3)

5.3.1.1 The Data Generating Process

Our model is based on the RBC model with oil due to Finn (2000), with two modifications. First, we assume that there is no link from capital utilization

\[ \alpha \] is the share of capital in production.
Second, we assume that the model is driven by multiple shocks. This is to ensure that the covariance matrix of the data for key endogenous variables is non-singular, thus allowing us to consider a systems estimation of a misspecified model in Section 5.3.1.2. Since this model is well documented in Finn (2000), we only provide a sketch of the model, focusing on the innovations.

The representative consumer solves:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[ \gamma_t \ln c_t^* + \eta s_t^* \ln (1 - h_t^*) \right], \quad (5.4)$$

subject to

$$c_t^* + k_{t+1}^* - (1 - \delta s_t^*) k_t^* = A_t^* (k_t^* u_t^*)^\alpha (h_t^*)^{1-\alpha} - p_t^* e_t^*, \quad (5.5)$$

where

$$e_t^* = \frac{\nu_0}{v_1} (u_t^*)^{v_2} k_t^*, \quad (5.6)$$

and we use the asterix superscript (*) to denote endogenous variables in the data generating process.

The utility function in (5.4) is defined over consumption, $c^*$, and hours worked, $h^*$, which are measured relative to an endowment of leisure time that is normalized to unity. The utility function also contains two preference shocks, $\gamma_t$ and $s_t^*$, which follow stochastic processes that are described below. The resource constraint (5.5) states that expenditures on consumption and investment (the net change in the capital stock $k^*$) must be financed by output, less expenditure on energy, $e^*$. Output is determined by a Cobb-Douglas production function with inputs of labor ($h^*$) and capital services, defined as the product of the physical capital stock and the utilization rate ($u^*$). The resource constraint contains two shocks: a conventional TFP shock ($A^*$) and a shock to the depreciation rate of capital ($s^*$). Finally, the demand

---

10So our version of the model is very similar to that analyzed by Sustek (forthcoming).
for energy is given by (5.6). This demand function is posited by Finn (2000) to summarize the assumption that higher rates of capital utilization require additional energy inputs.

The equations determining the equilibrium of this model are:

\[
\frac{\eta s_t^\eta}{1 - h_t^\eta} = (1 - \alpha) \frac{\gamma y_t^*}{c_t^* h_t^*},
\]

(5.7)

\[
\alpha \frac{y_t^*}{u_t^*} = p_t^e v_0 (u_t^*)^{\gamma h_t^*} k_t^*,
\]

(5.8)

\[
\frac{\gamma t}{c_t^*} = \beta E_t \gamma t+1 \left[ \alpha \frac{y_t^*}{k_t^*} + 1 - \delta^h s_t^h - p_t^e \frac{v_0}{v_1} (u_t^*)^{\gamma h_t^*} \right],
\]

(5.9)

\[
y_t^* = A_t^* (h_t^* (k_t^*)^{1-\alpha} (u_t^*)^\alpha),
\]

(5.10)

\[
y_t^* = c_t^* + k_t^* - (1 - \delta^h s_t^h) k_t^* + p_t^e \frac{v_0}{v_1} (u_t^*)^{\gamma h_t^*} k_t^*.
\]

(5.11)

We close the model with the assumption that the shocks follow univariate AR(1) processes with independent innovations:

\[
\ln p_t^e = \rho^e \ln p_{t-1}^e + \epsilon_t^e,
\]

\[
\ln A_t^* = \rho^A* \ln A_{t-1}^* + \epsilon_t^A*,
\]

\[
\ln \gamma_t = \rho^\gamma \ln \gamma_{t-1} + \epsilon_t^\gamma,
\]

\[
\ln s_t^\eta = \rho^s \ln s_{t-1}^\eta + \epsilon_t^\eta,
\]

\[
\ln s_t^h = \rho^s \ln s_{t-1}^h + \epsilon_t^h.
\]

The parameter values chosen for the model are given in Table 1, where \( \sigma \) is used to denote the standard deviations of the shocks \( \epsilon \).

Parameters \( \eta \) and \( v_0 \) are not shown because they do not appear in the log-linearized version of the model we use for the analysis. However, they are implicitly determined by our assumption that, in the steady state, hours worked as a share of total leisure time is one third and that energy expenditure is 5% of total output. This parametrization of the model generates data with qualitatively similar properties to US data. For example, the variances
of consumption and hours relative to output are 0.51 and 0.87, respectively. Gomme and Rupert (2007), Table 6, report values of 0.49 and 1.00 for US data. The correlations of output, consumption and hours with lagged output are 0.79, 0.60 and 0.70. This compares with the correlations of 0.84, 0.70 and 0.89 reported in Table 6 of Gomme and Rupert (2007). The model correlations are somewhat lower than those in US data, but the important stylized fact that consumption is less strongly correlated with lagged output than either hours or output is replicated quite well.

The (unconditional) variance decomposition of the data generated by this parametrization of the model is summarized in Table 2.

Two key points emerge from the variance decomposition. First, no single shock determines the majority of the variance of all endogenous variables (though the labor supply shock plays an important role). Second, the small variance of oil price shocks means that they contribute relatively little to the variance of the endogenous variables - around two or three per cent in general.

5.3.1.2 The Policymaker’s Model

We assume that the policymaker has access to a model that is similar to the actual data generating process, but misspecified in the sense that it includes no role for oil or utilization. We set up the policymaker’s model in a very similar way to the ”prototype economies” analyzed in the BCA literature. Specifically the model is one where agents solve:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t [\ln c_t + \eta \ln (1 - h_t)],$$

subject to

$$c_t + k_{t+1} - (1 - \delta) k_t = (1 - \tau^k_t) r_t k_t + (1 - \tau^h_t) w_t h_t + T_t.$$
The utility function is similar to the data generating process except that the preference shocks are absent. The resource constraint states that consumption and investment expenditure must be financed by income from supplying factors of production net of distortionary taxes ($\tau^k$ and $\tau^h$) and lump-sum transfers, $T$. The distortionary taxes are the wedges used in a BCA analysis. The returns from supplying capital and labor to firms are the rental rate $r$ and the wage rate $w$. The first-order conditions for this problem, combined with the first-order conditions for factor demands of a perfectly competitive firm, are:

\[ \frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} \left[ \alpha (1 - \tau^k_{t+1}) \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right], \quad (5.12) \]

\[ \frac{\eta}{1 - h_t} = (1 - \alpha) \frac{(1 - \tau^h_t) y_t}{c_t h_t}. \quad (5.13) \]

The market clearing conditions are:

\[ y_t = A_t h_t^{1-\alpha} k_t^\alpha \quad (5.14) \]

\[ y_t = c_t + k_{t+1} - (1 - \delta) k_t + g_t \quad (5.15) \]

where the final equation follows from the assumption that distortionary taxes net of transfers are used to finance government expenditure, $g$, through a balanced budget fiscal policy.

A BCA approach would close the model by assuming that the wedges $- \tau^k$, $\tau^l$, $A$, $g$ follow a flexibly-specified VAR. However, we assume that the policymaker treats the model as a DSGE model so that the wedges are assumed to follow AR(1) processes with orthogonal innovations:

\[ \tau^h_t = \rho_h \tau^h_{t-1} + (1 - \rho_h) \tau^h + \varepsilon^h_t, \]

\[ \ln A_t = \rho_A \ln A_{t-1} + \varepsilon^A_t, \]

\[ \ln g_t = \rho_g \ln g_{t-1} + (1 - \rho_g) g + \varepsilon^g_t, \]
\[ \tau_t^k = \rho_k \tau_{t-1}^k + (1 - \rho_k) \tau_t^k + \varepsilon_t^k. \]

In summary, the policymaker’s model consists of equations (5.12), (5.13), (5.14), and (5.15), specifying the behavior of four endogenous variables – \( k, h, c, y \) – which are driven by four shocks to \( \tau^k, \tau^l, A, \) and \( g \).

As noted above, we assume that the policymaker has access to a very long run of data for consumption, hours and output from the data generating process. These data allow the policymaker to calibrate the parameters that pin down the first moments of the data. Specifically, the policymaker calibrates the parameters as described in Table 3. \( \psi_g \) is the share of government expenditure in output.

The parameters governing the dynamics of the model are chosen in order to match the autocovariances of the data for consumption, output and hours. Specifically, let \( \Gamma_j \) denote the autocovariance matrix of these data at horizon \( j \) (this is computed from the asymptotic moments of the data generating process). Similarly, denote \( \hat{\Gamma}_j(\theta) \) as the autocovariance matrix implied by the policymaker’s model for parameter values \( \theta \). To proxy the policymaker’s estimation exercise, we choose the parameters of the policymaker’s model to minimize the distance between the autocovariances of consumption, output and hours from the model and that observed in the data. That is

\[ \hat{\theta} = \arg \min_\theta \left( \hat{\xi}(\theta) - \xi \right)' \left( \hat{\xi}(\theta) - \xi \right) \]

where \( \hat{\xi}(\theta) \equiv \text{vec} \left[ \hat{\Gamma}_0(\theta) \ \hat{\Gamma}_1(\theta) \ \hat{\Gamma}_2(\theta) \ \hat{\Gamma}_3(\theta) \ \hat{\Gamma}_4(\theta) \right] \) and analogously for \( \xi \).

The AR(1) coefficients for government spending and depreciation are not particularly intuitive, but these shocks are also associated with very small variances, suggesting that productivity and labor tax generate most of the variance in the policymaker’s model.

Figure 1 depicts the autocovariance functions for the data generating
process (solid blue line) and the policymaker’s model (dashed red line). The figure shows that the policymaker’s model matches the autocovariances of the data reasonably well, though with relatively less success for consumption. This is most likely driven by the lack of (positive) serial correlation in the investment wedge shock process entering the Euler equation in the policymaker’s model.

5.3.1.3 Adding the Missing Channel

In this section, we follow the approach outlined in Section 5.2.3 to add oil prices to the policymaker’s model. We augment the policymaker’s model with equations mapping oil price shocks to "factors" that feed into the structural shocks as follows:

\[ \tau^h_t = \tilde{\tau}^h_t + F^h_t, \]
\[ \ln \left[ \frac{A_t}{A} \right] = \ln \left[ \frac{\tilde{A}_t}{A} \right] + F^A_t, \]
\[ \ln \left[ \frac{g_t}{g} \right] = \ln \left[ \frac{\tilde{g}_t}{g} \right] + F^g_t, \]
\[ \tau^k_t = \tilde{\tau}^k_t + F^k_t, \]
\[ F^h_t = \Phi^h_F F^h_{t-1} + \Psi^h \varepsilon^e_t, \]
\[ F^A_t = \Phi^A_F F^A_{t-1} + \Psi^A \varepsilon^e_t, \]
\[ F^g_t = \Phi^g_F F^g_{t-1} + \Psi^g \varepsilon^e_t, \]
\[ F^k_t = \Phi^k_F F^k_{t-1} + \Psi^k \varepsilon^e_t, \]
\[ \ln p^e_t = \rho^e \ln p^e_{t-1} + \varepsilon^e_t. \]

We assume that the policymaker estimates the parameters \( \vartheta = \{ \Phi_i, \Psi_i \} \) for \( i = h, A, g, k \), by matching the coefficients of a VAR(1) model estimated on data for consumption, hours, output and oil prices. We choose this approach because it is a straightforward way of estimating model parameters that could be used in practical applications. We assume that the long data series allows the policymaker to compute the true value of \( \rho^e \) and the variance of the oil
price shock directly from the data.

Denoting the VAR coefficient matrices from the data generating process as $C$ and $C(\vartheta)$ as the corresponding coefficients implied by the policymaker’s model, we find the coefficients that solve:

$$\hat{\vartheta} = \arg\min_{\vartheta} \left( \text{vec} \left[ C(\vartheta) \right] - \text{vec} \left[ C \right] \right)^\prime \left( \text{vec} \left[ C(\vartheta) \right] - \text{vec} \left[ C \right] \right).$$

The resulting coefficients are recorded in Table 5.

With these parameter values in hand, we can compute the impulse response to an oil price shock in the policymaker’s augmented model and compare it to the true impulse response from the data generating process. These are shown in Figure 2 as dashed red lines and solid blue lines, respectively. We see that the policymaker’s augmented model generates a reasonable match to the true impulse response function, with the exception of consumption. This is consistent with the observation noted above that the policymaker’s model finds it difficult to fit the behavior of consumption.

### 5.3.2 Missing House Prices

In this section, we consider a model where the missing channel is more deeply embedded within the endogenous structure of the economy. Specifically, we assume that the data generating process is a model where there is an important role for house prices in determining consumption. We follow the same steps as in Section 5.3.1: specifying the data generating process; specifying the policymaker’s (misspecified) model and fitting it to the data generating process; and finally augmenting the policymaker’s model to try to account for the missing channel.

#### 5.3.2.1 The Data Generating Process

We use the model of Iacoviello (2005). Here, we simply provide a sketch of the model, as the original paper contains a clear and detailed exposition. The
model is a variant of the Bernanke et al. (1999) New Keynesian model where endogenous changes in the balance sheets of firms create a financial accelerator effect. The model also includes collateral constraints tied to the value of commercial property for firms and a fraction of the credit constrained households are assumed to hold nominal debt. These features create a financial accelerator where demand shocks are amplified. When demand rises, asset prices rise, which in turn increases the borrowing capacity of debtors. This boosts consumption spending and investment. As consumer prices rise, the real value of debtors’ outstanding obligations falls and real net worth rises. Because borrowers have a higher propensity to spend than lenders, there are further increases in demand.

As noted, the main innovations of the model are related to the behavior of demand. The remainder of the model is standard. Calvo price setting leads to a conventional New Keynesian Phillips curve relating inflation to marginal costs. The monetary policymaker is assumed to operate a reaction function for the nominal interest rate, which has a Taylor (1993) formulation adjusted to include interest rate smoothing. The model is driven by four shocks: to technology, the Phillips curve, monetary policy rule and housing preferences. The housing preference shock is a stochastic variation in the relative weight on housing in consumers’ utility functions. We refer to this shock as a house price shock (following Iacoviello, 2005) in what follows.

Iacoviello (2005) sets the parameters of the model using a minimum distance estimator that matches the impulse responses of the model to those in an identified VAR estimated on US data. For our data generating process, we use the parameter values reported in the paper. Importantly, the parametrization implies that house price shocks are relatively variable and persistent compared to the other shocks in the model.
5.3.2.2 The Policymaker’s Model

We assume that the policymaker uses a very standard New Keynesian model:

\[ x_t = E_t x_{t+1} - \sigma (R_t - E_t \pi_{t+1} - r^*_t), \]
\[ \pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t, \]
\[ R_t = \phi \pi_t + \gamma x_t + \varepsilon_t, \]

where \( x \) is the output gap, \( \pi \) is quarterly inflation and \( R \) is the quarterly nominal interest rate. The shock processes for the natural real interest rate (\( r^* \)) and the cost-push and policy shocks (\( u \) and \( \varepsilon \)) follow simple AR(1) processes:

\[ r^*_t = \rho_r r^*_{t-1} + \eta^r_t, \]
\[ u_t = \rho_u u_{t-1} + \eta^u_t, \]
\[ \varepsilon_t = \rho_\varepsilon \varepsilon_{t-1} + \eta_\varepsilon^t. \]

As in Section 5.3.1.2, we match the policymaker’s model to the data generating process using a minimum distance estimator.\(^\text{11}\) We choose to calibrate the discount factor at \( \beta = 0.99 \) and estimate the remaining parameters. The parameter values that minimize the distance between the autocovariance functions of the model and data generating process are contained in Table 6. The coefficient on the real interest rate in the IS curve (\( \sigma \)) and the slope of the Phillips curve (\( \kappa \)) are both relatively high in comparison to many DSGE based estimates. The data generating process suggests that output is quite volatile relative to inflation and nominal interest rates. To match this, the policymaker’s model requires aggregate demand to be very sensitive to changes in the real interest rate.

\(^{11}\)We assume that the output gap data available to the policymaker correspond to the log-deviation of output from steady state generated by the true model. This approximation seems reasonable given that the parameter values estimated by Iacoviello (2005) imply a minor role of technology shocks.
5.3. **TWO SIMPLE EXAMPLES**

Figure 3 demonstrates that the autocovariance functions are matched reasonably well, with the exception of the nominal interest rate. One notable feature of the results in Figure 3 is that the autocovariance matrix of the policymaker’s model is symmetric. This follows from the fact that the model contains just three state variables (corresponding to the structural shocks) which are each driven by an independent stochastic process. This property of the model makes it harder for it to match the autocovariance properties of the data generating process which contains a richer array of state variables.\(^\text{12}\)

5.3.2.3 **Adding the Missing Channel**

To incorporate the effect of house prices, we once more assume that the structural shock process evolves according to

\[
\begin{align*}
  s_t &= \tilde{s}_t + \Lambda F_t, \\
  \tilde{s}_t &= B\tilde{s}_{t-1} + \varepsilon_t, \\
  F_t &= \Phi F_{t-1} + \Psi u_t, \\
  p^h_t &= J p^h_{t-1} + u_t.
\end{align*}
\]

However, in this application, we assume that the autoregressive matrix of the factors (\(\Phi\)) is full, to capture the rich correlation structures that can be expected to be generated by a channel such as the financial accelerator mechanism in the Iacoviello (2005) model. To economize on parameters, we assume that there are just two factors driving the three structural shocks.

\(^{12}\)Another, simple explanation for the failure to match the autocovariances of the nominal interest rate stems from the fact that we use an unweighted distance metric and nominal interest rates have the smallest variance in the data generating process. Scaling the distance metric appropriately (perhaps by the (inverse) variance of the relevant series) should mitigate this effect.
That is, we have

\[ F_t = \begin{bmatrix} F_{1t} \\ F_{2t} \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \]

\[ \Lambda = \begin{bmatrix} \Lambda^r_1 & \Lambda^r_2 \\ \Lambda^u_1 & \Lambda^u_2 \\ \Lambda^\varepsilon_1 & \Lambda^\varepsilon_2 \end{bmatrix}. \]

As before, we assume that consistent estimates of \( J \) and the variance of \( u \) are directly obtained from the data. The remaining parameters are estimated by minimizing the quadratic distance between the coefficients of the VAR(1) for the output gap, inflation, the nominal interest rate and house prices implied by the data generating process and the VAR(1) coefficients implied by the policymaker’s model. One difference with the approach used in Section 5.3.1.3 is that we allow the variances of the structural shocks to be re-estimated. The reason is that the data generating process suggests that house price shocks drive a relatively high proportion of output variations. This means that the initial estimates of the shock processes presented in Section 5.3.2.2 are picking up a large contribution from house price shocks. Holding these parameters fixed when trying to estimate the factor structure implies that there is ”too much” variance in the model (when house price shocks \( u \) are also included) which makes it difficult to fit the VAR(1) coefficients accurately.

The parameter estimates are presented in Table 7. We see that the estimation process pushes the variance of the shock to the IS curve towards zero. This is consistent with the fact that house price shocks play an important role in driving the data generating process. The two factors that map house price shocks into structural shocks are both reasonably persistent and correlated.

The resulting impulse responses are depicted in Figure 4. The augmented DSGE model matches the impulse responses reasonably well, with the ex-
5.3. TWO SIMPLE EXAMPLES

ception of the nominal interest rate response. This result is consistent with the findings of Section 5.3.1.3 and suggests that estimates of the impulse responses of variables that are not well matched to the data (in terms of their autocovariance properties in our case) may be unreliable.

5.3.3 Discussion

The examples in Sections 5.3.1 and 5.3.2 are in many ways very benign cases of misspecification. The policymaker’s model is a relatively good approximation of the true data generating process. And the policymaker has access to an infinite sample of data from that data generating process. Even so, the model does a relatively poor job in matching some impulse responses to oil or house price shocks. In particular, one may be suspicious of the impulse responses constructed for variables that fit the data relatively poorly in the baseline specification of the DSGE model. This observation prompts several points that are worthy of discussion.

In a more realistic setting, it seems likely that small sample issues will interfere with the accuracy with which one can uncover the true impulse response using the methodology outlined above. Indeed, the empirical exercises in Caldara and Harrison (2008) indicate that it may be extremely difficult to uncover statistically significant relationships between missing channel proxies and the state variables for a medium-scale DSGE model using US data. Given the intricacies involved in fitting the augmented misspecified model to the data, an obvious question is why not simply use a VAR model to capture the desired impulse response function?

In particular, a VAR model could be specified that includes the variables in the DSGE model plus missing channel proxies. Identification assumptions could then be imposed on the VAR to identify a shock to the missing channel variable. The resulting impulse response would provide an estimate of the likely impact of the shock of interest on the variables included in the DSGE model. This VAR impulse response can be used directly to inform forecast
and policy discussions. In this approach, there is no need to work with the misspecified benchmark DSGE model or adapt it in any way. This is certainly an attractive alternative. One problem, however, is that in a realistic application, the size of the VAR that we use is likely to only contain a subset of the endogenous variables in the baseline DSGE model. Another problem is that the identification scheme used to isolate the shock of interest may not be obvious or uncontroversial, particularly if the VAR is (relatively) large.

As noted above, the key issue appears to be accurately extracting the relevant information about impulse response functions using the types of methods described above. Indeed, for our simple examples, our methodology linking structural shocks to missing channel proxies works very well if the true impulse response is known. Specifically, for the example considered in Section 5.3.1, it is possible to match the true oil price shock using our ‘atheoretical factor’ approach by minimizing a quadratic criterion function defined in terms of the deviations in the impulse responses from the augmented baseline model and the true model. Figure 5 demonstrates this result. Figure 6 reports the same experiment for the example considered in Section 5.3.2. Based on these results, our empirical exercise in Section 5.4 shows how a VAR may be used to identify the impulse response that we wish to match.

### 5.4 Empirical Application

The discussion in Section 5.3.3 suggests that DSGE and VAR models could be used eclectically to investigate particular channels that are not included in the DSGE model. In this section, we assume that the policymaker uses the Smets and Wouters (2007) model, but is interested in the implications of alternative assumptions about the future path of house prices for the variables in the model. Although the Smets and Wouters (2007) model includes a wide

---

13 Using a FAVAR model might be a useful way if mitigating some aspects of the dimensionality issue.
5.4. EMPIRICAL APPLICATION

range of frictions and transmission channels, it does not include the housing market. Therefore, we use a small VAR to help us adjust the baseline DSGE model projections in the light of alternative house price scenarios.

We proceed as follows. In Section 5.4.1, we briefly describe the Smets and Wouters (2007) model and the US data set that we assume to be available for the forecaster. In Section 5.4.2, we describe the VAR that is used to identify the effects of house price shocks on a small number of key macroeconomic variables. In Section 5.4.3 we use the methodology described in Section 5.2 to incorporate shocks to house prices into the DSGE model. This is done by augmenting the processes that describe the structural shocks of the DSGE model to include a missing channel proxy (house prices) and parametrization of the augmented processes so that the response of the DSGE model to a shock to housing prices matches that of the identified VAR. In Section 5.4.4, we use the augmented model to investigate how an alternative scenario for house prices alters the projections of the variables in the DSGE model.

5.4.1 The DSGE Model and Data Set

We use the medium-scale DSGE model of Smets and Wouters (2007). As noted in the Introduction, this model has been used as a blueprint for the operational DSGE models developed at a number of central banks. It is also becoming an important benchmark model in the literature. Given that the model is very well known, we only provide a sketch of its structure.

The model includes a wide variety of nominal and real frictions. Households maximize utility subject to habit formation in their consumption choices. They accumulate capital (which they rent to firms) subject to costs of adjusting the rates of investment and utilization. Households (via unions) also supply differentiated labor to firms and set the nominal wage according to a Calvo scheme. Wages that are not re-optimized are increased in line with a weighted average of trend nominal wage growth and lagged inflation.

Firms rent capital services and labor from households which are used
to produce output. Output is used for consumption, government purchases and investment. Retailers set prices according to a Calvo mechanism, with a partial indexation of prices that are not reoptimised that is analogous to the scheme for nominal wages described above. Monetary policy is operated through a reaction function for the nominal interest rate. The reaction function is such that nominal interest rates respond to deviations of inflation from the target, the output gap and the rate of change of the output gap. The output gap is defined using a flexible price specification of the model.

The model is driven by seven shocks: to the level of TFP; to the investment adjustment cost function; to household preferences; to government spending; to price and wage mark-ups; and to the monetary policy reaction function. Government spending and TFP shocks are assumed to be correlated. These shocks are designed to explain the movements of seven data series: GDP growth; consumption growth; investment growth; inflation (GDP deflator); the Fed funds rate; real wage growth; and hours worked.

Smets and Wouters (2007) estimate the parameters of the model using Bayesian maximum likelihood. We use the modal parameter estimates reported in the paper for our empirical exercise. We use the Kalman filter to construct an estimate of the state vector at the final data point which is then projected using the state space representation of the model to produce the forecasts.

We construct an extended version of the US data set used by Smets and Wouters (2007). Our data set is almost identical to theirs for the common period, indicating that there have only been minor revisions to the backdata.\footnote{Data and model code for the Smets and Wouters (2007) exercise is available from the AER website.} Our data set runs from 1947–2008, but we assume that the forecaster only has a shorter data set. This allows us to use the remaining data to assess the forecasts generated from the model under alternative assumptions about house prices, in Section 5.4.4.
5.4.2 The VAR Model

We construct a small VAR along the lines of that estimated by Iacoviello (2005). We use the output, inflation and interest rate data from the data set described in Section 5.4.1. Following Iacoviello (2005), we use the Conventional Mortgage Home Price index to construct a measure of house prices.

We estimate a VAR(2) over the period 1975Q1–2004Q4 and identify a house price shock using a Cholesky decomposition with the following ordering: nominal interest rate, inflation, house prices and output. This ordering follows Iacoviello (2005) and generates a broadly similar impulse response function, depicted in Figure 7.\(^{15}\)

5.4.3 Incorporating House Price Effects into the DSGE Model

Now we use the methodology outlined in Section 5.2 to match the VAR house price shock using the DSGE model. Our approach is to map the house price shock into a subset of the structural shocks in the DSGE model. We assume house price shocks to be manifested through correlated movements in three structural shocks: the household preference shock; the price mark-up shock and the monetary policy shock. The two latter shocks are motivated by the fact that inflation and nominal interest rates appear in the VAR and these shocks have the most direct effect on them. While it is output rather than consumption that appears in the VAR, we choose to map house price shocks into preference shocks because this type of mapping is implied by models that include a household financial accelerator such as Aoki et al. (2004) and Iacoviello (2005).

As in the example from Section 5.3.2.3, we choose to map two factors into

\(^{15}\)This should be compared with Figure 1 of Iacoviello (2005). We do not expect the impulse responses to be identical as there are some minor differences in data series and sample sizes.
the three structural shocks of the DSGE model. So we assume that

\[
F_t = \begin{bmatrix} F_{1t} \\ F_{2t} \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}
\]

\[
\Lambda = \begin{bmatrix} \Lambda^b_1 & \Lambda^b_2 \\ \Lambda^p_1 & \Lambda^p_2 \\ \Lambda^r_1 & \Lambda^r_2 \end{bmatrix}
\]

where \( \Lambda^b_i \), \( \Lambda^p_i \) and \( \Lambda^r_i \) denote the loadings on factor \( i (=1,2) \) for the preference shock, the price mark-up shock and the interest rate reaction function shock.

We also assume that the (log) relative house price evolves according to a simple AR(2) process:

\[
p^h_t = \phi_1 p^h_{t-1} + \phi_2 p^h_{t-2} + s_t \varepsilon_t,
\]

where \( \varepsilon_t \) is an iid Gaussian disturbance with unit variance.

We minimize the distance between the impulse response generated by the DSGE model and the identified VAR discussed in Section 5.4.2. Table 8 reports the resulting parameter estimates.

Figure 8 plots the impulse response from the augmented DSGE model and identified VAR, verifying that the match is reasonably close. The figure also includes the impulse responses of the DSGE observables that are not included in the VAR. This reveals that the house price shock behaves rather like a demand shock. An increase in house prices boosts consumption and investment. There is an expansion in output, thus inducing an increase in real wages and hours worked.

### 5.4.4 Alternative House Price Scenario

In this section, we consider synthetic forecast simulations that involve alternative assumptions about the future path of house prices. We assume that the forecaster has access to data up until 2006Q4, which represents the peak
5.4. EMPIRICAL APPLICATION

We first construct a baseline projection from the DSGE model. This projection is produced using the Smets and Wouters (2007) model (using the modal parameter values reported in the paper) by using the Kalman filter to estimate the state vector of the DSGE model and projecting it using the transition equation.

We then construct a baseline projection for house prices. For this purpose, we use the equation for house prices from the empirical VAR, but assume that the other variables (inflation, output and the Fed Funds rate) evolve as forecasted in the DSGE model. This gives us a projection for house prices that is consistent with the DSGE projection for other variables. The VAR equation for house prices is very persistent (the maximum eigenvalue of the coefficients on lagged house prices is very close to unity). This means that the house price projection is one where relative house prices are expected to remain at an elevated level for a prolonged period of time. Naturally, the actual path of relative house prices was much weaker than this projection.

This allows us to construct an alternative scenario for house prices that is more in line with the actual data. For this purpose, we suppose that the forecaster wishes to investigate how the DSGE model projections would change if house prices were materially weaker than the projection produced from the VAR. We assume that the path chosen by the forecaster lies halfway between the VAR forecast and data. This proxies a case where the forecaster wants to examine a substantial shock, but does not have perfect foresight of the actual path of house prices.

To impose this scenario on our DSGE projection, we assume that the lower path of house prices is solely generated by a sequence of house price shocks. We find the sequence of house price shocks that delivers the alternative house price projection. Then, we apply these shocks to the augmented DSGE model described in Section 5.4.3 under the assumption that they are anticipated (so that they are part of the information set used to produce the
projection). To apply the anticipated shocks, we solve the augmented model using the AIM algorithm developed by Anderson and Moore (1985). This solution algorithm expresses the solution for the endogenous variables of the model in the form

\[ x_t = B x_{t-1} + \sum_{j=0}^{\infty} F^j \Gamma \epsilon_{t+j}, \]

which allows us to easily compute the solution of the model by inserting the relevant sequence of shocks \( \epsilon \).

The resulting projections are shown in Figure 9. In each panel, the blue line represents the data that we assume to be available to the forecaster. The green line represents the subsequent data outturns. The red dashed line is the baseline DSGE forecast and the black dot-dash line is the forecast augmented for the house price scenario described above. Growth rates are shown as four-quarter growth rates.

Figure 9 shows that the DSGE forecasts that are consistent with the alternative house price scenario are weaker than the baseline forecasts. The lower path for house prices is interpreted as a negative demand shock. For some forecast profiles (investment, real wages), the alternative scenario brings the projections more in line with the actual realizations. But in most cases, the baseline projections appear to be more accurate.

It is impossible to draw conclusions about the efficacy of our approach from one example. But it is tempting to conclude that in this case, the house price scenario worsens the forecasts from the DSGE model. It seems likely that a key challenge is to identify the implicit projection for house prices that is consistent with the baseline DSGE forecast: it is possible that this projection looks much more like the actual house price profile that we have assumed. Our approach was to use the simple VAR model from Section 5.4.2. But that approach ignores valuable information contained in the baseline DSGE projection. A promising alternative approach would be to construct the baseline house price projection using the methodology proposed by Schorfheide et al.
Another issue is that the way in which we map the missing channel into the DSGE model requires a degree of judgment. We assumed consumption to be the key demand component that is affected by house price shocks. While this is in line with some theoretical models, we know in practice that dwellings investment was particularly affected by the collapse of US house prices over the 2007–08 period. An alternative approach would therefore be to map house price shocks to investment adjustment costs in the DSGE model. Initial experiments with this approach indicate that it may be difficult to identify this mapping if investment does not appear in the VAR that is used to generate the target impulse responses.

5.5 Conclusions

In this paper we consider the problem of how to analyze the effects of shocks that do not appear explicitly within a DSGE model that is used to inform policy and forecast discussions. We believe that this is an important problem to address, despite the increasing size and scope of operational DSGE models at use in policy institutions. We argue that insights from the business cycle accounting literature – namely that missing channels may manifest themselves in terms of correlated shocks to a misspecified model – should be incorporated into the DSGE forecasting toolkit. We propose a simple and flexible method to approximate the effects of shocks that are not incorporated in the baseline forecast model. An application of the method will typically require judgment over precisely how it should be implemented. The development of models that incorporate some of the channels currently missing from operational forecast models would be a welcome aid to that judgment.
Bibliography


TAYLOR, J. B., “Discretion versus policy rules in practice,” Carnegie-
Rochester Conference Series on Public Policy 39 (December 1993), 195–
214.

A Appendix

A.1 Details of Example from Section 5.3

A.1.1 BCA Equivalence

Here, we briefly consider the business cycle accounting equivalence between
the policymaker’s model and the data generating process. Inspection of the
first-order conditions implies that the equilibrium sequences are equivalent
when the shocks in the policymaker’s model satisfy:

\[(1 - \tau_t^h)^{-1} = \eta s_t^n / \gamma_t,\]
\[A_t = A_t^* \left( u_t^* \right)^{\alpha},\]
\[g_t = \delta \left( s_t^d - 1 \right) k_t^* + \beta_t^{v_0} (u_t^*)^{v_1} k_t^*,\]
\[\tau_t^k = 1 - \frac{\gamma_t + 1}{\gamma_t} \left[ \alpha y_t^* / k_t^* + 1 - \delta s_t^d - \beta_t^{v_0} (u_t^*)^{v_1} \right] + 1 - \delta.\]

A.1.2 Steady State

The equations describing the steady state of the policymaker’s model are:

\[\frac{k}{y} = \alpha \left[ \beta^{-1} - 1 + \delta \right]^{-1},\]
\[1 = \frac{c}{y} + \delta \frac{k}{y} + \frac{g}{y}.\]
Hence:

\[
\frac{c}{y} = \left[ 1 - \delta \alpha \left( \beta^{-1} - 1 + \delta \right)^{-1} - \psi_g \right],
\]

where \(\psi_g\) is the share of government spending in output. Moreover:

\[
\frac{\eta}{1 - h} = (1 - \alpha) \frac{1 - \tau^h y}{ch},
\]

\[
y = h^{1-\alpha} k^\alpha. \tag{5.17}
\]

The equations describing the steady state of the data generating process are:

\[
y = c + \delta k + p^e v_0 \frac{v_0}{v_1} u^1 k,
\]

\[
= c + \delta k + \psi_e y,
\]

where \(\psi_e\) is the share of energy expenditure in output (and can be calibrated by the appropriate choice of \(v_0\) given steady-state allocations for \(u\) and \(k\), the steady-state oil price and the parameter \(v_1\)). Moreover, the steady state of the consumption Euler equation is:

\[
1 = \beta \left[ \alpha \frac{y}{k} + 1 - \delta - \psi_e \frac{y}{k} \right],
\]

\[
= \beta \left[ (\alpha - \psi_e) \frac{y}{k} + 1 - \delta \right],
\]

which means that \(\frac{k}{y} = \frac{\alpha - \psi_e}{\beta^{1-1+\delta}}\) and \(\frac{c}{y} = 1 - \frac{\delta(\alpha - \psi_e)}{\beta^{1-1+\delta}} - \psi_e\).

### A.1.3 Log-linearized Equations

The log linearized equations of the policymaker’s model are:

\[
\dot{c}_t = \dot{c}_{t+1} - \left[ 1 - \beta (1 - \delta) \right] \left( \dot{y}_{t+1} + \dot{y}_{t+1} - \dot{k}_{t+1} \right), \tag{5.18}
\]
\[
\frac{1}{1-h} \dot{h}_t = \dot{y}_t - \dot{c}_t - \tau^h, \quad (5.19)
\]
\[
\dot{y}_t = \bar{A}_t + (1-\alpha) \dot{h}_t + \alpha \dot{k}_t, \quad (5.20)
\]
\[
\dot{y}_t = \left[1 - \frac{\delta \alpha}{\beta^{-1} - 1 + \delta} - \psi_g\right] \dot{c}_t + \frac{\alpha}{\beta^{-1} - 1 + \delta} \dot{k}_{t+1}, \quad (5.21)
\]
\[
- \frac{(1-\delta) \alpha}{\beta^{-1} - 1 + \delta} \dot{k}_t + \psi_g \dot{g}_t,
\]

where \(\psi_g\) is the size of the government spending wedge relative to output in the steady state.

The log-linearized equations of the data generating process are:

\[
\frac{1}{1-h} \dot{h}^*_t + \dot{s}^*_t = \dot{y}^*_t - \dot{c}^*_t + \dot{\gamma}_t, \quad (5.22)
\]
\[
\dot{y}^*_t - \dot{u}^*_t = \bar{p}^*_t + (v_1 - 1) \dot{u}^*_t + \dot{\kappa}^*_t, \quad (5.23)
\]
\[
\dot{\gamma}_t - \dot{c}^*_t = E_t (\dot{\gamma}_{t+1} - \dot{c}^*_t) + \frac{\alpha (1-\beta (1- \delta))}{\alpha - \psi_e} E_t (\dot{y}^*_t - \dot{\kappa}^*_{t+1}), \quad (5.24)
\]
\[
- \beta \delta \dot{s}^*_t = \psi_e (1 - \beta (1 - \delta)) \frac{1}{\alpha - \psi_e} E_t (\dot{p}^*_t - v_1 \dot{u}^*_t),
\]
\[
\dot{y}^*_t = \bar{A}^*_t + (1-\alpha) \dot{h}^*_t + \alpha \dot{k}^*_t + \alpha \dot{u}^*_t, \quad (5.25)
\]
\[
\dot{g}^*_t = \left[1 - \frac{\delta (\alpha - \psi_e)}{\beta^{-1} - 1 + \delta} - \psi_e\right] \dot{c}^*_t + \frac{\alpha - \psi_e}{\beta^{-1} - 1 + \delta} \dot{k}^*_{t+1}, \quad (5.26)
\]
\[
- \left[\frac{(1-\delta) (\alpha - \psi_e)}{\beta^{-1} - 1 + \delta} - \psi_e\right] \dot{k}^*_t,
\]
\[
+ \frac{\delta (\alpha - \psi_e)}{\beta^{-1} - 1 + \delta} \dot{s}^*_t - \psi_e [\bar{p}^*_t + v_1 \dot{u}^*_t],
\]

where \(\psi_e\) is the steady-state expenditure on energy as a share of output.
A.2 Tables and Figures

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.99</td>
<td>$\alpha$</td>
<td>0.36</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>$\nu_1$</td>
<td>2</td>
</tr>
<tr>
<td>$\rho_e$</td>
<td>0.99</td>
<td>$\sigma_e$</td>
<td>0.0675</td>
</tr>
<tr>
<td>$\rho_{A^*}$</td>
<td>0.95</td>
<td>$\sigma_A$</td>
<td>0.12</td>
</tr>
<tr>
<td>$\rho_\eta$</td>
<td>0.85</td>
<td>$\sigma_\eta$</td>
<td>0.25</td>
</tr>
<tr>
<td>$\rho_\delta$</td>
<td>0.1</td>
<td>$\sigma_\delta$</td>
<td>0.85</td>
</tr>
<tr>
<td>$\rho_\gamma$</td>
<td>0.9</td>
<td>$\sigma_\gamma$</td>
<td>0.175</td>
</tr>
</tbody>
</table>

Table 1: Parameters for data generating process – oil model

<table>
<thead>
<tr>
<th>Variable</th>
<th>Contribution to variance from</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon^\gamma$</td>
</tr>
<tr>
<td>Output</td>
<td>0.04</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.24</td>
</tr>
<tr>
<td>Hours</td>
<td>0.05</td>
</tr>
<tr>
<td>Utilization</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 2: Variance decomposition for data generating process – oil model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.99</td>
<td>$\alpha$</td>
<td>0.31</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>$\psi_g$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 3: Parameters of the policymaker’s model – oil model example
### Table 4: Forcing process parameters of policymaker’s model – oil model example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_A$</td>
<td>0.70</td>
<td>$\sigma_A$</td>
<td>0.26</td>
</tr>
<tr>
<td>$\rho_h$</td>
<td>0.85</td>
<td>$\sigma_h$</td>
<td>0.27</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>-0.94</td>
<td>$\sigma_g$</td>
<td>0.0062</td>
</tr>
<tr>
<td>$\rho_k$</td>
<td>-0.98</td>
<td>$\sigma_k$</td>
<td>0.0042</td>
</tr>
</tbody>
</table>

### Table 5: Estimated factor loadings for policymaker’s augmented model – oil model example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_h$</td>
<td>0.16</td>
<td>$\Phi_h$</td>
<td>0.85</td>
</tr>
<tr>
<td>$\Psi_A$</td>
<td>-1.03</td>
<td>$\Phi_A$</td>
<td>0.93</td>
</tr>
<tr>
<td>$\Psi_g$</td>
<td>-57.78</td>
<td>$\Phi_g$</td>
<td>0.46</td>
</tr>
<tr>
<td>$\Psi_k$</td>
<td>38.48</td>
<td>$\Phi_k$</td>
<td>-0.15</td>
</tr>
</tbody>
</table>

### Table 6: Estimated parameters of the policymaker’s model – house prices example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>4.37</td>
<td>$\rho_u$</td>
<td>0.68</td>
<td>$\lambda_1^e$</td>
<td>0.73</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.45</td>
<td>$\rho_\varepsilon$</td>
<td>0.33</td>
<td>$\lambda_2^e$</td>
<td>0.81</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1.75</td>
<td>$\text{Var}(\eta_t^r)$</td>
<td>0.01$^2$</td>
<td>$\text{Var}(\eta_t^u)$</td>
<td>0.70$^2$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.00</td>
<td>$\text{Var}(\eta_t^u)$</td>
<td>0.70$^2$</td>
<td>$\text{Var}(\eta_t^e)$</td>
<td>0.66$^2$</td>
</tr>
</tbody>
</table>

### Table 7: Estimated parameters for ‘missing house price’ process

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_1$</td>
<td>-0.02</td>
<td>$\Phi_{22}$</td>
<td>0.59</td>
<td>$\lambda_1^e$</td>
<td>0.73</td>
</tr>
<tr>
<td>$\Psi_2$</td>
<td>0.30</td>
<td>$\lambda_1^r$</td>
<td>-0.26</td>
<td>$\lambda_2^e$</td>
<td>0.81</td>
</tr>
<tr>
<td>$\Phi_{11}$</td>
<td>0.91</td>
<td>$\lambda_2^r$</td>
<td>0.73</td>
<td>$\text{Var}(\eta_t^r)$</td>
<td>$10^{-8}$</td>
</tr>
<tr>
<td>$\Phi_{12}$</td>
<td>-0.03</td>
<td>$\lambda_1^u$</td>
<td>-0.84</td>
<td>$\text{Var}(\eta_t^u)$</td>
<td>0.94$^2$</td>
</tr>
<tr>
<td>$\Phi_{21}$</td>
<td>0.68</td>
<td>$\lambda_2^u$</td>
<td>-1.35</td>
<td>$\text{Var}(\eta_t^e)$</td>
<td>0.02$^2$</td>
</tr>
</tbody>
</table>
Table 8: Estimated parameters for house price shock in Smets and Wouters model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_1$</td>
<td>0.09</td>
<td>$\Phi_{22}$</td>
<td>0.13</td>
<td>$\Lambda_1^r$</td>
<td>1.22</td>
</tr>
<tr>
<td>$\Psi_2$</td>
<td>1.51</td>
<td>$\Lambda_1^b$</td>
<td>0.51</td>
<td>$\Lambda_2^r$</td>
<td>0.96</td>
</tr>
<tr>
<td>$\Phi_{11}$</td>
<td>0.68</td>
<td>$\Lambda_2^b$</td>
<td>−0.62</td>
<td>$\phi_1$</td>
<td>1.51</td>
</tr>
<tr>
<td>$\Phi_{12}$</td>
<td>−0.41</td>
<td>$\Lambda_1^p$</td>
<td>−1.71</td>
<td>$\phi_2$</td>
<td>−0.52</td>
</tr>
<tr>
<td>$\Phi_{21}$</td>
<td>1.41</td>
<td>$\Lambda_2^p$</td>
<td>0.38</td>
<td>$s_h$</td>
<td>0.78</td>
</tr>
</tbody>
</table>

Figure 1: Autocovariances from data generating process (blue solid line) and policymaker’s model (red dashed line).
Figure 2: Impulse responses to oil price shock: data generating process (blue solid line) and policymaker’s model augmented for oil price effects (red, dashed line).
Figure 3: Autocovariance functions of data generating process (blue, solid) and policymaker’s model (red, dashed).
Figure 4: Impulse responses to house price shock from data generating process (blue, solid) and augmented baseline model (red, dashed).
Figure 5: Impulse responses from data generating process (blue, solid) and augmented baseline model (red, dashed) to oil price shock when true impulse response is known.
Figure 6: Impulse response of data generating process (blue, solid) and augmented baseline model (red, dashed) to house price shock when true impulse response is known.
Figure 7: House price shock responses from empirical VAR.
Figure 8: Response to house price shock in augmented Smets Wouters model (blue) and identified VAR (red dashed).

Figure 9: Forecasts from baseline DSGE model (red, dashed) and house price scenario (black, dot-dashed) versus data out-turns (green, solid).
MONOGRAPH SERIES

3. Hamilton, Carl B. Project Analysis in the Rural Sector with Special Reference to the Evaluation of Labour Cost, 1974
5. Myhrman, Johan Monetary Policy in Open Economies, 1975
7. Wihlborg, Clas Capital Market Integration and Monetary Policy under Different Exchange Rate Regimes, 1976
10. Calmfors, Lars Prices, Wages and Employment in the Open Economy, 1978
11. Kornai, János Economics of Shortage, Vols I and II, 1979

239


17. Sellin, Peter *Asset Pricing and Portfolio Choice with International Investment Barriers*, 1990


25. Daltung, Sonja *Risk, Efficiency, and Regulation of Banks*, 1994


27. Stennek, Johan *Essays on Information-Processing and Competition*, 1994


29. Dahlquist, Magnus *Essays on the Term Structure of Interest Rates and Monetary Policy*, 1995


37. Domeij, David *Essays on Optimal Taxation and Indeterminacy*, 1998
38. Flodén, Martin *Essays on Dynamic Macroeconomics*, 1999
43. Johansson, Åsa *Essays on Macroeconomic Fluctuations and Nominal Wage Rigidity*, 2002
44. Groth, Charlotta *Topics on Monetary Policy*, 2002
45. Gancia, Gino A. *Essays on Growth, Trade and Inequality*, 2003
47. Kohlscheen, Emanuel *Essays on Debts and Constitutions*, 2004
49. Stavlö, Ulrika *Essays on Culture and Trade*, 2005
50. Herzing, Mathias *Essays on Uncertainty and Escape in Trade Agreements*, 2005
52. Pienaar, Natalie *Economic Applications of Product Quality Regulation in WTO Trade Agreements*, 2005
54. Eisensee, Thomas *Essays on Public Finance: Retirement Behavior and Disaster Relief*, 2005

55. Favara, Giovanni *Credit and Finance in the Macroeconomy*, 2006


57. Larsson, Anna *Real Effects of Monetary Regimes*, 2007


60. Queijo von Heideken, Virginia *Essays on Monetary Policy and Asset Markets*, 2007


64. Damsgaard, Erika Färnstrand *Essays on Technological Choice and Spillovers*, 2008


66. Folke, Olle *Parties, Power and Patronage: Papers in Political Economics*, 2010


70. Perrotta, Maria Carmela *Aid, Education and Development*, 2010